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NUMERICAL ENCLOSURE OF SOLUTIONS FOR TWO DIMENSIONAL DRIVEN CAVITY PROBLEMS

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Abstract. *In this paper, we consider a numerical enclosure method for stationary solutions of two dimensional regularized driven cavity problems. The infinite dimensional Newton method takes an important role in our method, which needs to estimate the rigourous bound for the norm of inverse of the linearized operator. The method can be applied to the case for the large Reynolds numbers. Numerical examples which show the actual usefullness of the method are presented.*

1 Introduction

We consider the following steady state and homogeneous Navier-Stokes equations

$$\begin{cases} -\Delta u + R \cdot (u \cdot \nabla) u + \nabla p &= 0 \text{ in } \Omega, \\ \text{div } u &= 0 \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where u, p and R are the velocity vector, pressure and the Reynolds number, respectively and the flow region Ω is a unit square $(0,1)\times(0,1)$ in $\mathbf{R^2}$. In what follows, for each rational number m, let $H^m(\Omega)$ denote the L^2-Sobolev space of order m on Ω. We suppose that the function $g = (g_1, g_2)$ satisfies $g \in H^{1/2}(\partial\Omega, \mathbf{R^2})$. In the classical driven cavity problem, we sometimes meet the irregular boundary conditions such that $g \notin H^{1/2}(\partial\Omega, \mathbf{R^2})$ for which the problem (1.1) has no H^1 solution(*cf.* [9]). In this paper, in order to avoid such a difficulty, we only treat a kind of regularized problem. Namely, we assume that there exists a function $\varphi \in H^2(\Omega)$ satisfying $(\varphi_y, -\varphi_x) = g$ on $\partial\Omega$. And, particularly, for comparison with the result obtained by Wieners [9], in our numerical examples we consider the case of $\varphi(x,y) = x^2(1-x)^2y^2(1-y)$, though our method can also be applied to more general Navier -Stokes problems. In [9], the numerical enclosure of the problem (1.1) was studied for the small Reynolds numbers based on Plum's method(e.g., [5]) incorporating with the Newton-Kantorovich theorem. His method, however, can not be applied to the large Reynolds numbers, because the verification condition could not be satisfied at all for the large R because of the dependence on the Reynolds number. In the present paper, we also use the Newton-like verification condition, but our method has an advantage that it can also be applied to large Reynolds numbers, provided that the approximation space is sufficiently accurate and that the exact inverse operator actually exists in the rigorous sense.

In the following section, first, we translate the problem (1.1) into the stream function formulation and introduce the linearized operator. Next, we present a numerical verification method to assure the invertibility of the linearized operator as well as show a method to estimate an upper bound of the norm of the inverse operator. An infinite dimensional Newton's method to prove numerically the existence of solutions for the original nonlinear problem is derived in Section 3, and, finally, we will give some numerical results in Section 4.

2 Invertibility of the linearized operator

The incompressibility condition in (1.1) admits us to introduce a stream function ψ satisfying $u = (\frac{\partial \psi}{\partial y}, \ -\frac{\partial \psi}{\partial x})$. Using this relation we can rewrite the equations (1.1) as

$$\begin{cases} \Delta^2\psi + R\cdot J(\psi,\Delta\psi) &= 0 \ \text{in}\Omega, \\ \psi &= \varphi \ \text{on } \partial\Omega, \\ \frac{\partial \psi}{\partial n} &= \frac{\partial \varphi}{\partial n} \ \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where J is a bilinear form defined by $J(u,v) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$ and $\frac{\partial}{\partial n}$ means the normal derivative. Further, newly setting u as $\psi - \varphi$, we have

$$\begin{cases} \Delta^2 u + \Delta^2\varphi + R\cdot J(u+\varphi,\Delta(u+\varphi)) &= 0 \ \ \text{in}\Omega, \\ u = \frac{\partial u}{\partial n} &= 0 \ \ \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Our aim is to verify the existence of a weak solution $u \in H_0^2(\Omega)$ of (2.2), where $H_0^2(\Omega) \equiv \{v \in H^2(\Omega) \mid v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\}$, and we adopt the inner product by $< u, v >_{H_0^2} \equiv (\Delta u, \Delta v)_{L^2}$ for $u, v \in H_0^2(\Omega)$, and the norm is defined by $\|u\|_{H_0^2} \equiv \|\Delta u\|_{L^2}$ for $u \in H_0^2(\Omega)$, where $(\cdot,\cdot)_{L^2}$ and $\|\cdot\|_{L^2}$ mean the inner product and the norm on $L^2(\Omega)$, respectively.

In what follows, let S_h be the set of bicubic C^2-spline functions on Ω with uniform and rectangular partition of the mesh size h (e.g., [7]).

We first enclose an approximate solution $u_h \in S_h$ of (2.2) satisfying

$$(\Delta u_h + \Delta\varphi, \Delta v_h)_{L^2} + (R\cdot J(u+\varphi,\Delta(u_h+\varphi)), v_h)_{L^2} = 0 \qquad \text{for all } v_h \in S_h. \tag{2.3}$$

Then the linearized operator at u_h in weak sense is represented as follows:

$$\mathcal{L}u \equiv \Delta^2 u + R\cdot\{J(u_h+\varphi,\Delta u) + J(u,\Delta(u_h+\varphi))\}.$$

Defining the canonical scalar products we have

$$\begin{aligned} < \Delta^2 u, v >_{H^{-2},H_0^2} &\equiv (\Delta u, \Delta v)_{L^2}, \\ < J(u,\Delta u), v >_{H^{-2},H_0^2} &\equiv (J(v,u),\Delta u)_{L^2}, \end{aligned}$$

then $\mathcal{L}$ is considerd as the operator from $H_0^2(\Omega)$ to $H^{-2}(\Omega)$. Our first aim is to verify the existence of the inverse operator $\mathcal{L}^{-1} : H^{-2}(\Omega) \to H_0^2(\Omega)$ and, next, to formulate the infinite dimensional Newton method for the nonlinear problem (2.2).

By direct computations, we find that for any $q \in H^{-2}(\Omega)$ there exists a unique solution $v \in H_0^2(\Omega)$ satisfying

$$\begin{cases} \Delta^2 v &= q \ \ \text{in } \Omega, \\ v = \frac{\partial v}{\partial n} &= 0 \ \ \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Now, for each $q \in H^{-2}(\Omega)$, let Kq be the unique solution $v \in H_0^2(\Omega)$ of the equation (2.4), then $K : H^{-1}(\Omega) \to H_0^2(\Omega)$ is compact. Using the following compact operator on $H_0^2(\Omega)$

$$F_1(u) \equiv -R \cdot K\{J(u_h + \varphi, \Delta u) + J(u, \Delta(u_h + \varphi))\},$$

the equation $\mathcal{L}u = 0$ is equivalent to the fixed point equation

$$u = F_1(u). \tag{2.5}$$

Thus, owing to the linearity of the operator and the Fredholm alternative, in order to verify the invertibility of the operator $\mathcal{L}$, we only have to show the uniqueness of the solution of (2.5).
Now let $P_h : H_0^2(\Omega) \longrightarrow S_h$ denote the H_0^2-projection defined by

$$(\Delta(u - P_h u), \Delta v_h)_{L^2} = 0 \qquad \text{for all } v_h \in S_h,$$

and we consider the constructive error estimations for P_h. At first, we obtain the following interpolation error estimates(*cf.* [7]). In the present paper, we omitted the proofs of the lemmas and theorems, which will be described in the forthcoming paper [1].
Lemma 1. *Let $\mathcal{I}_\Omega$ denote the bicubic spline interpolation oprator on Ω. For any $u \in H^4(\Omega) \cap H_0^2(\Omega)$ we have*

$$\|u - \mathcal{I}_\Omega u\|_{H_0^2} \le 2\frac{h^2}{\pi^2}\|\Delta^2 u\|_{L^2}. \tag{2.6}$$

Using Lemma 1, the property

$$\|u - P_h u\|_{H_0^2} = \inf_{\xi \in S_h} \|u - \xi\|_{H_0^2} \le \|u - \mathcal{I}_\Omega u\|_{H_0^2}$$

and some duality arguments, we have the following error estimates for P_h.
Lemma 2. *For $u \in H^4(\Omega) \cap H_0^2(\Omega)$ we have*

$$\|u - P_h u\|_{H_0^2} \le 2\frac{h^2}{\pi^2}\|\Delta^2 u\|_{L^2}, \tag{2.7}$$

$$\|u - P_h u\|_{H_0^1} \le \sqrt{8}\frac{h^3}{\pi^3}\|\Delta^2 u\|_{L^2}, \tag{2.8}$$

$$\|u - P_h u\|_{L^2} \le 4\frac{h^4}{\pi^4}\|\Delta^2 u\|_{L^2}. \tag{2.9}$$

Now, as in [2] or [4], we decompose (2.5) into the finite and infinite dimensional parts, and apply a Newton-like method only for the finite dimensional part, which leads to the following operator:

$$\mathcal{N}_h^1(u) \equiv P_h u - [I - F_1]_h^{-1}(P_h u - P_h F_1(u)),$$

where I is the identity map on $H_0^2(\Omega)$. And we assumed that the restriction to S_h of the operator $P_h[I - F_1] : S_h \to S_h$ has the inverse $[I - F_1]_h^{-1}$. The validity of this assumption can be numerically checked in the actual computations.

We next define the compact operator $T_1 : H_0^2(\Omega) \longrightarrow H_0^2(\Omega)$ by

$$T_1(u) \equiv \mathcal{N}_h^1(u) + (I - P_h)F_1(u),$$

then we have the following equivalence relation

$$u = T_1(u) \Longleftrightarrow u = F_1(u).$$

Then, our purpose is to find a unique fixed point of T_1 in a certain set $U \subset H_0^2(\Omega)$ which is called a 'candidate set'. Given positive real numbers γ and α we define the corresponding candidate set U by

$$U \equiv U_h \oplus [\alpha], \tag{2.10}$$

where $U_h \equiv \{\phi_h \in S_h \mid \|\phi_h\|_{H_0^2} \le \gamma\}$, $[\alpha] \equiv \{\phi_\perp \in S_\perp \mid \|\phi_\perp\|_{H_0^2} \le \alpha\}$ and $S_\perp$ means the orthogonal complement of S_h in $H_0^2(\Omega)$. If the relation

$$\overline{T_1(U)} \subset \mathrm{int}(U) \tag{2.11}$$

holds, then by Schauder's fixed point theorem and by the linearity of T_1, there exists a fixed point u of T_1 in U and the fixed point is unique, i.e., $u = 0$, which implies that the operator $\mathcal{L}$ is invertible. Decomposing (2.11) into finite and infinite dimensional parts we have a sufficient condition for (2.11) as follows:

$$\begin{cases} \sup_{u \in U} \|\mathcal{N}_h^1(u)\|_{H_0^2} < \gamma \\ \sup_{u \in U} \|(I - P_h)F_1(u)\|_{H_0^2} < \alpha. \end{cases} \tag{2.12}$$

Now, by some arguments using the error estimations in Lemma 2, we have the following theorem which yields a omputable sufficient condition for the verification condition (2.12).

Theorem 1. *Let $\{\phi_i\}_{1\le i\le N}$ be a basis of S_h and define the following constants:*

$$\begin{aligned}
C_1 &= \|\nabla(u_h+\varphi)\|_\infty,\ C_2 = \left\|\nabla\frac{\partial(u_h+\varphi)}{\partial x}\right\|_\infty + \left\|\nabla\frac{\partial(u_h+\varphi)}{\partial y}\right\|_\infty \\
C_3 &= \|\nabla\Delta(u_h+\varphi)\|_\infty,\ C_p = \frac{1}{\pi\sqrt{2}},\ M_1 = \|L^T G^{-1} L\|_E, \\
K_1 &= 2R\frac{h}{\pi}C_1 + 4R\frac{h^3}{\pi^3}C_3, \\
K_2 &= 2R\frac{h}{\pi}C_1 + 2\sqrt{2}R\frac{h^2}{\pi^2}C_3C_p, \\
K_3 &= \sqrt{2}RM_1\frac{h}{\pi}(2C_1 + C_2C_p) + 2\sqrt{2}RM_1\frac{h^2}{\pi^2}C_3C_p,
\end{aligned}$$

where $\|\cdot\|_E$ denotes the matrix norm corresponding to the Euclidian vector norm in $\mathbf{R}^N$, C_p is the Poincaré constant, the N dimensional matrix $G = (G_{ij})$ is defined by

$$G_{ji} \equiv R(J(u_h+\varphi, \Delta\phi_i) + J(\phi_i, \Delta(u_h+\varphi)), \phi_j)_{L^2} + (\Delta\phi_i, \Delta\phi_j)_{L^2},$$

and $D = LL^T$ is a Cholesky decomposition for the matrix $D = (D_{ij})$ defined by

$$D_{ij} \equiv (\Delta\phi_i, \Delta\phi_j)_{L^2}.$$

For the above constants K_1, K_2 and K_3, if the inequality

$$K_1 + K_2K_3 < 1 \tag{2.13}$$

holds then the operator $\mathcal{L}$ is invertible.

Next, we have the following estimation of the norm for the inverse of the linearized operator, which plays an essential role to realize the Newton-like method for the verification of the nonlinear problem (2.2).

Theorem 2. *Assume that the invertibility condition (2.13) holds. Then using the same constants in Theorem 1, it follows that*

$$M_2 \equiv \|\mathcal{L}^{-1}\|_{B(H^{-2},H_0^2)} \ \le\ \sqrt{\left(\frac{K_2M_1+1}{1-K_1-K_2K_3}\right)^2 + \left(\frac{K_3(K_2M_1+1)}{1-K_1-K_2K_3} + M_1\right)^2}.$$

3 Verification procedure for driven cavity flows

In this section, we assume that the invertibility of the linearized operator $\mathcal{L}$ is validated by the method described in the previous section. As usual, e.g., [2], [3], [4], we will verify the existence of solutions for (2.2) in the neighborhood of $\bar{u}$ satisfying

$$\left\{ \begin{array}{rcll} \Delta^2 \bar{u} & = & -\Delta^2 \varphi - R \cdot J(u_h + \varphi, \Delta(u_h + \varphi)) & \text{in } \Omega, \\ \bar{u} = \frac{\partial \bar{u}}{\partial n} & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{3.1}$$

Note that $u_h = P_h \bar{u}$. Defining $v_0 \equiv \bar{u} - u_h$, we see that $v_0 \in S_\perp$ and, by the similar arguments deriving Lemma 2, the error estimates for v_0 can be obtained as follows:

$$\begin{array}{rcl} \|v_0\|_{H_0^2} & \leq & 2\dfrac{h^2}{\pi^2} \| - \Delta^2 \varphi - R \cdot J(u_h + \varphi, \Delta(u_h + \varphi))\|_{L^2}, \\ \|v_0\|_{H_0^1} & \leq & \sqrt{2}\dfrac{h}{\pi} \|v_0\|_{H_0^2}, \\ \|v_0\|_{L^2} & \leq & 2\dfrac{h^2}{\pi^2} \|v_0\|_{H_0^2}. \end{array}$$

Writing $w = u - \bar{u}$ and defining the following compact map on $H_0^2(\Omega)$

$$F_2(w) \equiv RK\{J(u_h + \varphi, \Delta(u_h + \varphi)) - J(w + u_h + v_0 + \varphi, \Delta(w + u_h + v_0 + \varphi))\}, \tag{3.2}$$

we have the fixed point equation

$$w = F_2(w), \tag{3.3}$$

which is equivalent to (2.2).

Now we formulate the infinite dimensional Newton method for the equation (3.3). We define the compact operator $T_2(w) \equiv \mathcal{L}^{-1} q(w)$ in $H_0^2(\Omega)$, where

$$q(w) \equiv R\{J(u_h + \varphi, \Delta(u_h + \varphi)) - J(w + u_h + v_0 + \varphi, \Delta(w + u_h + v_0 + \varphi)) \\ + J(u_h + \varphi, \Delta w) + J(w, \Delta(u_h + \varphi))\}.$$

Then we have the relation

$$w = F_2(w) \Longleftrightarrow w = T_2(w). \tag{3.4}$$

We intend to find a fixed point of T_2 in a set W defined by

$$W = \{w \in H_0^2(\Omega) \mid \|w\|_{H_0^2} \leq \alpha\}, \tag{3.5}$$

where α is a positive number. If the relation

$$T_2(W) \subset W \tag{3.6}$$

holds, then by Schauder's fixed point theorem there exists a fixed point of T_2 in W. Since a sufficient condition for (3.6) is

$$\sup_{w\in W} \|T_2(w)\|_{H_0^2} \le \alpha, \tag{3.7}$$

by estimating the left-hand side of (3.7), we obtain the following numerical condition for the verification of solutions of the nonlinear problem (2.2).

Theorem 3. *Assume that the invertibility condition (2.13) holds. Using the same constants as in Theorem 1 and 2, and defining the constants:* $b \equiv \|v_0\|_{H_0^2}$, $C_4 \equiv \frac{1}{\pi}$, *if there exists a real number* $\alpha > 0$ *satisfying*

$$M_2 R \left\{ \sqrt{2} C_p C_1 b + 2\sqrt{2} C_p C_3 b \frac{h^2}{\pi^2} + C_4^2 (\alpha + b)^2 \right\} \le \alpha, \tag{3.8}$$

then there exists a fixed point of T_2 *in* W. *Here, the constant* C_4 *comes from the embedding estimates of the form* $\|\nabla u\|_{L^4} \le C_4 \|\Delta u\|_{L^2}$ *for* $u \in H_0^2(\Omega)$ [8].

4 Numerical examples

In calculations, we used the interval arithmetic in order to avoid the effects of rounding errors in the floating-point computations. All computations were carried out on the DELL Precision WorkStation 650 (Intel Xeon 3.2GHz) using MATLAB (Ver. 6.5.1) and the interval arithmetic toolbox INTLAB (Ver. 4.2.1) coded by Prof. Rump in TU Hamburg-Harburg ([6]). The verification results are shown in Table 1, in which 'smallest α' means the smallest bound α satisfing the verification condition (3.8) and the solution u in (2.2) is enclosed as $\|u - u_h\|_{H_0^2(\Omega)} \le \|v_0\|_{H_0^2(\Omega)} + \alpha$.

Due to the computational cost, the mesh size $h = 1/23$ was the practical limit of our computing system using interval arithmetic. For your reference, we illustrated the result, in Table 2, where all computations were performed by using the usual floating point arithmetic of double precision.

It seems that Wieners' method can not be applied to the Reynolds number larger than $R = 20$ in [9]. On the other hand we enclosed the stationary solution for the Reynolds number over 130. As shown in Table 2, our method can be applied, in principle, to any large Reynolds numbers, if we can use more accurrate approximation subspaces, i.e., smaller mesh size.

Table 1: Verification Results for Driven Cavity Problem ($h = 1/23$)

R	M_1	M_2	C_1	C_2	C_3	$\|v_0\|_{H_0^2}$	smallest α
100	1.0429	1.9845	0.0625	0.7328	2.7149	1.6991e-3	2.3619e-3
110	1.0448	2.1856	0.0625	0.7342	2.7223	1.7099e-3	3.0394e-3
120	1.0467	2.4296	0.0625	0.7356	2.7299	1.7216e-3	4.0491e-3
130	1.0487	2.7315	0.0625	0.7371	2.7376	1.7343e-3	5.9244e-3
135	1.0497	2.9109	0.0625	0.7378	2.7416	1.7409e-3	8.4399e-3

Table 2: Verification Results for Driven Cavity Problem ($h = 1/50$)

R	M_1	M_2	C_1	C_2	C_3	$\|v_0\|_{H_0^2}$	smallest α
200	1.0346	1.7958	0.0625	0.7496	3.0410	3.8670e-4	3.9306e-4

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