A COMPUTATIONAL APPROACH TO CONSTRUCTIVE
A PRIORI AND A POSTERIORI ERROR ESTIMATES
FOR FINITE ELEMENT APPROXIMATIONS
OF BI-HARMONIC PROBLEMS

Mitsuhiro T. Nakao
Faculty of Mathematics, Kyushu University
Hakozaki, Fukuoka, 812-8581 Japan
(mtnakao@math.kyushu-u.ac.jp)

Kouji Hashimoto
Graduate School of Informatics, Kyoto University
Yoshida, Kyoto, 606-8501 Japan
(hashimot@amp.i.kyoto-u.ac.jp)

Kaori Nagatou
Faculty of Mathematics, Kyushu University
Hakozaki, Fukuoka, 812-8581 Japan
(nagatou@math.kyushu-u.ac.jp)

Abstract. In the numerical verification method of solutions for nonlinear fourth order elliptic equations, it is important to find a constant in the constructive a priori and a posteriori error estimates for the finite element approximation of bi-harmonic problems. We show these procedures by verified computational techniques using the Hermite spline functions for two dimensional rectangular domain. Several numerical examples which confirm the actual effectiveness of the method are presented.


1 Introduction

In this paper, we consider the guaranteed error bounds of the finite element approximations for the following equation:

\[
\begin{align*}
\Delta^2 u &= f \quad \text{in } \Omega, \\
u &= \partial_n u = 0 \text{ on } \partial\Omega,
\end{align*}
\]

where \( f \in L^2(\Omega) \) with a rectangular domain \( \Omega \) in \( \mathbb{R}^2 \), and \( \partial_n u \) denotes the outer normal derivative of \( u \).

1.1 Notations

In the below, setting \( \Omega := (0,1)^2 \) for simplicity, we denote the \( L^2 \) inner product on \( \Omega \) by \( (\cdot, \cdot)_{L^2} \) and the norm by \( \| \cdot \|_{L^2} \). We also denote the usual \( k \)-th order \( L^2 \) Sobolev space on \( \Omega \) by \( H^k(\Omega) \) for any positive integer \( k \) as well as the space \( H^2_0(\Omega) \) by

\[
H^2_0(\Omega) := \{ v \in H^2(\Omega) : v = \partial_n v = 0 \text{ on } \partial\Omega \}.
\]

Let \( S_h \subset H^2_0(\Omega) \), \( n = \dim S_h \), be a finite element space, under some rectangular mesh, which is spanned by two dimensional Hermite spline functions \( \{\phi_i\}_{1 \leq i \leq n} \) with homogeneous boundary conditions in \( H^2_0(\Omega) \) sense [4]. Moreover, we define the space \( S^*_h \), \( n^* = \dim S^*_h \), which is spanned by the basis \( \{\phi^*_i\}_{1 \leq i \leq n^*} \), as the finite element subspace of \( H^2(\Omega) \), not of \( H^2_0(\Omega) \), satisfying \( S_h \subset S^*_h \) but \( S_h \neq S^*_h(n < n^*) \). Namely, the set of base functions \( \{\phi^*_i\}_{1 \leq i \leq n^*} \) consists of the elements in \( \{\phi_i\}_{1 \leq i \leq n} \) and the Hermite spline functions corresponding to the boundary nodes (cf. [5]).

Next, we define the \( H^2_0 \)-projection \( P_h : H^2_0(\Omega) \rightarrow S_h \) of \( v \in H^2_0(\Omega) \) by

\[
(\Delta v - \Delta P_h v, \Delta \phi_h)_{L^2} = 0, \quad \forall \phi_h \in S_h.
\]

Moreover, we also define the \( L^2 \)-projection \( P_0 : L^2(\Omega) \rightarrow S^*_h \) of \( v \in L^2(\Omega) \) by

\[
(v - P_0 v, \phi^*_h)_{L^2} = 0, \quad \forall \phi^*_h \in S^*_h.
\]

Let \( S^x_h \) and \( S^y_h \) denote the set of one dimensional Hermite spline functions on \((0,1)\) with homogeneous \( H^2_0 \) boundary conditions in \( x^- \) and \( y^- \) directions, respectively. Then, \( S_h \) is represented as the tensor product \( S^x_h \otimes S^y_h \). Similarly, we have \( S^*_h = S^x_h \otimes S^y_{n^*} \), \( S_{n^*}^x \), \( S_{n^*}^y \) are spaces of one dimensional spline functions without boundary functions. In what follows, a parameter \( h \) stands for the maximum mesh size of the partition of the interval \((0,1)\).

In the \( x^- \)-direction, for \( v \in H^2_0(0,1) \) and \( w \in L^2(0,1) \), we also define the projections \( P^x_2 : H^2_0(0,1) \rightarrow S^x_h \) and \( P^x_0 : L^2(0,1) \rightarrow S^x_{n^*} \) by

\[
(D^2_{xx}w - D^2_{xx}P^x_2 w, D^2_{xx}\varphi_h)_{L^2} = 0, \quad \forall \varphi_h \in S^x_h,
\]

and

\[
(w - P^x_0 w, \varphi^*_h)_{L^2} = 0, \quad \forall \varphi^*_h \in S^x_{n^*},
\]

respectively. For \( P^y_2 \) and \( P^y_0 \), analogously defined in the \( y^- \)-direction.
1.2 Motivation

Let $u_h \in S_h$ be an approximate solution of (1.1) satisfying

$$\langle \Delta u_h, \Delta \phi_h \rangle_{L^2} = \langle f, \phi_h \rangle_{L^2} \quad \forall \phi_h \in S_h.$$ 

Then, note that we have $u_h = P_h u$ by the definition and that the solution $u$ of (1.1) belongs to $H^2_0(\Omega) \cap H^4(\Omega)$([1]). Therefore, in what follows, we will discuss on the error estimates for the projection operator $P_h$.

We now assume the following a priori error estimates.

**Assumption 1** For an arbitrary $v \in H^3_0(\Omega) \cap H^4(\Omega)$, there exists a constant $C_0$ such that

$$\|\Delta v - \Delta P_h v\|_{L^2} \leq C_0 h^2 \|\Delta^2 v\|_{L^2}.$$  

Our main purpose of this paper is to find an a priori constant $C_0$ in the assumption 1 by using guaranteed numerical computations on computer. And as a bi-product of the arguments, we also show a method to get an a posteriori error bound for the approximate solution of the equation (1.1). In the numerical verification method of solutions for two dimensional Navier-Stokes problems(e.g., [2]), we need to enclose a solution of bi-harmonic equations with guaranteed error bounds. In such a situation, the above constant and a posteriori error estimates for the finite element approximation play an essential and important role. The basic techniques used in the below are extension of the method in [5] or [3] to the bi-harmonic problem.

1.3 Preliminary results

We first introduce the following known results.

**Lemma 1** [4] For an arbitrary $\psi \in H^3_0(\Omega) \cap H^4(\Omega)$, it follows that

$$\|D^2_x \psi - D^2_x P^x_2 \psi\|_{L^2} \leq C h^2 \|D^2_x \psi\|_{L^2}, \tag{1.2}$$

where the constant $C$ can be taken as $C = 1/\pi^2$. Moreover, the estimate (1.2) is equivalent to the following inequality:

$$\|\psi - P^x_2 \psi\|_{L^2} \leq C h^2 \|D^2_x \psi - D^2_x P^x_2 \psi\|_{L^2}.$$ 

We now show the following inverse inequality for later use.

**Lemma 2** For $\psi_h \in S_h$, it follows that

$$\|D^2_x \psi_h\|_{L^2} \leq \frac{\kappa}{h^2} \|\psi_h\|_{L^2},$$

where $\kappa = 20\sqrt{21} < 91.6516$. 

141
Proof: Note that it is sufficient to prove the concerning inequality only for one-dimensional polynomial of degree 3 on the interval $[0, h]$. In [4], base functions for the $x$-direction on $[0, h]$ are given by
\[ ϕ_1(x) = (x - h)^2(2x + h)/h^3, \quad ϕ_2(x) = x^2(3h - 2x)/h^3, \]
\[ ϕ_3(x) = x(x - h)^2/h^2, \quad ϕ_4(x) = x^2(x - h)/h^2, \]
for $x \in [0, h]$. Hence, setting $ϕ_h := a_1ϕ_1(x) + a_2ϕ_2(x) + a_3ϕ_3(x) + a_4ϕ_4(x)$ and using some guaranteed computations of eigenvalue bounds of a matrix, we obtain
\[ \frac{∥D_2^2ϕ_h∥^2_{L^2(0,h)}}{∥ϕ_h∥^2_{L^2(0,h)}} \leq \frac{8400}{h^4}, \]
where $a = (a_1, a_2, a_3h, a_4h)^T$.

Here, in order to get the above bound, we used a direct calculation of the matrix eigenvalue. That is, denoting the first and the second matrices in the above by $A$ and $B$, respectively, let $B = D^TD$ a Cholesky decomposition of $B$. Then, it is readily seen that the maximum eigenvalue of the symmetric matrix $D^{-T}AD^{-1}$ presents a desired bound. By using a computer algebra system, we confirmed that it is equal to 10. Thus, we can take $κ$ as $κ = √8400 = 20√21$.

2 Main Results

In this section, we show the constructive a priori and a posteriori error estimations for approximate solutions of the equation (1.1), which are equivalent to the same error estimates for $H^2_0$-projection of the exact solution.

Let define
\[ v_h := P_h v = \sum_{i=1}^n v_i φ_i ∈ S_h \quad \text{and} \quad Δv_h := P_0Δv_h = \sum_{i=1}^{n^*} a_i^* φ_i^* ∈ S_{h^*}, \]
for an arbitrary $v ∈ H^2(Ω) ∩ H^4(Ω)$. Then, $Δv_h$ satisfies
\[ (Δv_h, φ_i^*)_{L^2} = -(∇v_h, ∇φ_i^*)_{L^2}, \quad ∀φ_i^* ∈ S_{h^*}. \]
And, for \( g \equiv \Delta^2 v \in L^2(\Omega) \), we set

\[
g_h \equiv \sum_{i=1}^{n} g_i \phi_i \in S_h,
\]

so that

\[
(g_h, \phi_h)_{L^2} = (g, \phi_h)_{L^2}, \quad \forall \phi_h \in S_h. \quad (2.2)
\]

Moreover, we denote some matrices and vectors as

\[
A_\ast = (A^*_i) = (\Delta \phi_j^*, \Delta \phi_i^*)_{L^2} \in \mathbb{R}^{n^* \times n^*}, \quad A = (A_{ij}) = (\Delta \phi_j, \Delta \phi_i)_{L^2} \in \mathbb{R}^{n \times n},
\]

\[
L_\ast = (L^*_i) = (\phi_j^*, \phi_i^*)_{L^2} \in \mathbb{R}^{n^* \times n^*}, \quad L = (L_{ij}) = (\phi_j, \phi_i)_{L^2} \in \mathbb{R}^{n \times n},
\]

\[
M = (M_{ij}) = (\nabla \phi_j, \nabla \phi_i^*)_{L^2} \in \mathbb{R}^{n^* \times n^*}, \quad N = (N_{ij}) = (\nabla \phi_j^*, \nabla \phi_i)_{L^2} \in \mathbb{R}^{n \times n^*}.
\]

\[
\bar{a} = (a_1, \ldots, a_n)^T \in \mathbb{R}^{n}, \quad \bar{v} = (v_1, \ldots, v_n)^T, \quad \bar{g} = (g_1, \ldots, g_n)^T \in \mathbb{R}^n.
\]

Notice that \( \|g\|_{L^2}^2 = \|g_h\|_{L^2}^2 + \|g - g_h\|_{L^2}^2 \). And, when we define the matrix \( Q \in \mathbb{R}^{n \times n} \) such that \( L = QQ^T \), it follows that \( \|Q^T \bar{g}\|_E = \|g_h\|_{L^2} \), where \( \| \cdot \|_E \) means the Euclidean norm in \( \mathbb{R}^n \). Under the above notations, the functions \( v_h \) and \( \Delta v_h \) are determined by solving the following matrix equations:

\[
A \bar{v} = L \bar{g},
\]

\[
L_\ast \bar{a} = -M \bar{v},
\]

respectively. Then, we have the following estimates.

**Lemma 3** It follows that

\[
\|\Delta v_h - \Delta \bar{v}_h\|_{L^2} \leq X \|g_h\|_{L^2},
\]

\[
\|g_h - \Delta \bar{v}_h\|_{L^2} \leq Y \|g_h\|_{L^2},
\]

where \( X \equiv \|Q^{-1}XQ^{-T}\|_E^{1/2}, \ Y \equiv \|Q^{-1}YQ^{-T}\|_E^{1/2} \). Here, setting \( Z \equiv L_\ast^{-1}MA^{-1}L \),

\[
X \equiv LA^{-1}L - Z^T L_\ast Z, \quad Y \equiv L - NZ - Z^T N^T + Z^T A_\ast Z.
\]

**Proof**: First, for the estimate \( \|\Delta v_h - \Delta \bar{v}_h\|_{L^2} \), we have

\[
\|\Delta v_h - \Delta \bar{v}_h\|_{L^2}^2 = \|\Delta v_h\|_{L^2}^2 - \|\Delta \bar{v}_h\|_{L^2}^2
\]

\[
= \bar{v}^T A \bar{v} - \bar{a}^T L_\ast \bar{a}
\]

\[
= \bar{v}^T A \bar{v} - \bar{v}^T M^T L_\ast^{-1} M \bar{v}
\]

\[
= \bar{g}^T (LA^{-1}L - LA^{-1}M^T L_\ast^{-1} M A^{-1} L) \bar{g}.
\]

143
Next, for the estimate $\|g_h - \Delta \Delta v_h\|_{L^2}$, it follows that
\[
\|g_h - \Delta \Delta v_h\|_{L^2}^2 \\
= (g_h, g_h)_{L^2} - (g_h, \Delta \Delta v_h)_{L^2} - (\Delta \Delta v_h, g_h)_{L^2} + (\Delta \Delta v_h, \Delta \Delta v_h)_{L^2} \\
= (g_h, g_h)_{L^2} + (\nabla g_h, \nabla \Delta v_h)_{L^2} + (\nabla \Delta v_h, \nabla g_h)_{L^2} + (\Delta \Delta v_h, \Delta \Delta v_h)_{L^2} \\
= g^T L \tilde{g} \tilde{g} + \tilde{g}^T N \tilde{a} + \tilde{a}^T N^T \tilde{g} + \tilde{a}^T A \tilde{a} \\
= g^T L \tilde{g} - \tilde{g}^T N L_{s^{-1}} M \tilde{v} - \tilde{v}^T M^T L_{s^{-1}} N^T \tilde{g} + \tilde{a}^T M^T L_{s^{-1}} A_s L_{s^{-1}} M \tilde{v} \\
= g^T (L - N L_{s^{-1}} M A^{-1} L - L A^{-1} M^T L_{s^{-1}} N^T + L A^{-1} M^T L_{s^{-1}} A_s L_{s^{-1}} M A^{-1} L) \tilde{g}.
\]
Thus by the above definitions of matrices $X, Y$ and $Z$, we have
\[
\|\Delta v_h - \Delta \Delta v_h\|_{L^2}^2 = g^T X \tilde{g}, \\
\|g_h - \Delta \Delta v_h\|_{L^2}^2 = g^T Y \tilde{g},
\]
which prove the lemma taking account that $\|Q^T \tilde{g}\|_E = \|g_h\|_{L^2}$.

\[\blacksquare\]

**Lemma 4** For an arbitrary $\psi \in H_0^2(\Omega)$, it follows that
\[
\|\Delta \psi - \Delta P^x_0 P^y_0 \psi\|_{L^2} \leq K \|\Delta \psi\|_{L^2},
\]
where $K = (2 + 2 (C \kappa + 1)^2)^{1/2}$, the constants $C$ and $\kappa$ are the same as in Lemma 1 and 2, respectively.

**Proof :** From the lemma 1 and 2, it follows that
\[
\|D_x^2 (\psi - P^x_2 \psi) - D_x^2 (P^x_0 \psi - P^x_0 P^y_0 \psi)\|_{L^2} \\
\leq \|D_x^2 (\psi - P^x_2 \psi)\|_{L^2} + \|D_x^2 (P^x_0 (\psi - P^x_0 \psi) )\|_{L^2} \\
\leq \|D_x^2 \psi\|_{L^2} + \frac{\kappa}{h^2} \|P^x_0 (\psi - P^x_2 \psi)\|_{L^2} \\
\leq \|D_x^2 \psi\|_{L^2} + \frac{\kappa}{h^2} \|\psi - P^x_2 \psi\|_{L^2} \\
\leq \|D_x^2 \psi\|_{L^2} + C h^2 \frac{\kappa}{h^2} \|D_x^2 \psi\|_{L^2} \\
\leq (C \kappa + 1) \|D_x^2 \psi\|_{L^2}.
\]
Thus, we have
\[
\|D_x^2 (\psi - P^x_0 P^y_0 \psi)\|_{L^2}^2 = \|D_x^2 (\psi - P^x_0 \psi) + D_x^2 (P^x_0 \psi - P^x_0 P^y_0 \psi)\|_{L^2}^2 \\
= \|D_x^2 (\psi - P^x_0 \psi)\|_{L^2}^2 + \|D_x^2 (P^x_0 (\psi - P^x_0 \psi) )\|_{L^2}^2 \\
\leq \|D_x^2 \psi\|_{L^2}^2 + \|D_x^2 (P^x_0 \psi - P^x_0 \psi)\|_{L^2}^2 \\
\leq \|D_x^2 \psi\|_{L^2}^2 + \|D_x^2 (\psi - P^x_2 \psi) - D_x^2 (P^x_0 \psi - P^x_0 \psi)\|_{L^2}^2 \\
\leq (1 + (C \kappa + 1)^2) \|D_x^2 \psi\|_{L^2}^2,
\]
where we have used the result just above to obtain the last right-hand side. Similarly, it follows that
\[
\|D_y^2 \psi - D_y^2 P^y_0 P^y_0 \psi\|_{L^2} \leq (1 + (C \kappa + 1)^2)^{1/2} \|D_y^2 \psi\|_{L^2}.
\]
Hence, we have the following estimation:

\[
\|\Delta\psi - \Delta P_0^h P_0^h \psi\|_{L^2} \leq \|D_2^2(\psi - P_0^h P_0^h \psi)\|_{L^2} + \|D_2^2(\psi - P_0^h P_0^h \psi)\|_{L^2} \\
\leq \left(1 + (C^2 K + 1)^2\right)^{1/2} (\|D_2^2\psi\|_{L^2} + \|D_2^2\psi\|_{L^2}) \\
\leq (2 + 2(CK + 1)^2)^{1/2} (\|D_2^2\psi\|_{L^2} + \|D_2^2\psi\|_{L^2})^{1/2} \\
\leq (2 + 2(CK + 1)^2)^{1/2}\|\Delta\psi\|_{L^2},
\]

where, in order to derive the last inequality, we have used the well known equality
\[
\|\Delta\psi\|^2_{L^2} = \|D_2^2\psi\|^2_{L^2} + \|D_2^2\psi\|^2_{L^2} + 2\|D_2^2\psi\|_{L^2} \text{ for any } \psi \in H_0^2(\Omega) \text{ on an arbitrary domain } \Omega. 
\]
Therefore, we obtain the constant \(K\) as in the lemma.

**Lemma 5** For an arbitrary \(\psi \in H_0^2(\Omega)\), it follows that

\[
\|\psi - P_0^h P_0^h \psi\|_{L^2} \leq C h^2 \|\Delta\psi\|_{L^2},
\]

where the constant \(C\) is the same as in Lemma 1.

**Proof**: From the lemma 1, it follows that

\[
\|\psi - P_0^h P_0^h \psi\|^2_{L^2} = \|\psi - P_0^h \psi + P_0^h \psi - P_0^h P_0^h \psi\|^2_{L^2} \\
= \|\psi - P_0^h \psi\|^2_{L^2} + \|P_0^h (\psi - P_0^h \psi)\|^2_{L^2} \\
\leq \|\psi - P_0^h \psi\|^2_{L^2} + \|\psi - P_0^h \psi\|^2_{L^2} \\
\leq C^2 h^4 \|D_2^2 \psi\|^2_{L^2} + C^2 h^4 \|D_2^2 \psi\|^2_{L^2} \\
\leq C^2 h^4 \|\Delta \psi\|^2_{L^2},
\]

which completes the proof.

Now, we show the following two main results of this paper.

**Theorem 1** (constructive a priori error estimates) The constant \(C_0\) in Assumption 1 can be taken as

\[
C_0 = C \cdot \left[(K X/(C h^2)) + Y\right]^2 + 1 \right)^{1/2},
\]

where the constants \(X, Y, C\) and \(K\) are defined in the previous lemmas 1, 3 and 4.

**Theorem 2** (a posteriori error estimates) For any \(v \in H_0^2(\Omega) \cap H^4(\Omega)\), let \(v_h := P_h v \in S_h\) and \(\Delta v_h := P_0 \Delta v_h \in S_h\). Then, it follows that

\[
\|\Delta v - \Delta v_h\|_{L^2} \leq K \|\Delta v_h - \Delta v_h\|_{L^2} + C h^2 \|\Delta^2 v - \Delta \Delta v_h\|_{L^2},
\]

where \(C\) and \(K\) are defined in the lemmas 1 and 4.
Theorem 1 by Table 1.

We first show several computational results for the constructive a priori constants in (3.2) respectively. Here, constants $\tilde{c}_h$ and $\tilde{c}_N$ of (1.1) is defined by

$$
\tilde{c}_h \equiv \frac{h^2}{2} \quad \text{and} \quad \tilde{c}_N \equiv \frac{h^2}{2}
$$

Thus, setting $\tilde{\psi} = v - \psi \in H^2_0(\Omega)$ and $\tilde{\psi}_0 \equiv P_0^T g \tilde{\psi} \in S_h$, from the lemmas 4 and 5, we obtain the desired estimates in Theorem 2.

Next, using Theorem 2, Lemma 3 and the property of the $L^2$-projection, it follows that

$$
\|\Delta v - \Delta v_h\|_{L^2} \leq K \|\Delta v_h - \Delta \tilde{v}_h\|_{L^2} + Ch^2 \|g - \Delta \tilde{v}_h\|_{L^2}
$$

which immediately completes the proof of Theorem 1.

3 Numerical examples

In this section, we present some numerical examples of a priori and a posteriori error estimates for the approximation of the bi-harmonic problem (1.1). Since the finite element solution $u_h \in S_h$ of (1.1) is defined by

$$
(\Delta u_h, \Delta \phi_h)_{L^2} = (f, \phi_h)_{L^2} \quad \forall \phi_h \in S_h,
$$

we have $u_h = P_h u$, and thus the above arguments can be applied to the error estimates for this approximate solution $u_h$. That is, using the procedure in the previous section to define $\tilde{\Delta} u_h \equiv P_h \Delta u_h \in S_h^*$, we obtain the a priori and a posteriori error estimates of the form

$$
\|\Delta u - \Delta u_h\|_{L^2} \leq C_0 h^2 \|f\|_{L^2},
$$

and

$$
\|\Delta u - \Delta u_h\|_{L^2} \leq K \|\Delta u_h - \tilde{\Delta} u_h\|_{L^2} + Ch^2 \|f - \Delta \tilde{\Delta} u_h\|_{L^2},
$$

respectively. Here, constants $C_0$, $K$ and $C$ are same as in theorems in Section 2.

We first show several computational results for the constructive a priori constants in Theorem 1 by Table 1.
Table 1: Numerical results for the a priori constant

<table>
<thead>
<tr>
<th>1/h</th>
<th>$C_0$</th>
<th>$X$</th>
<th>$Y$</th>
<th>$C_0h^2$</th>
<th>$C_0/C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.7225</td>
<td>3.6718e-4</td>
<td>1.7641</td>
<td>7.2256e-3</td>
<td>7.1314</td>
</tr>
<tr>
<td>20</td>
<td>0.7377</td>
<td>9.3350e-5</td>
<td>1.8257</td>
<td>1.8443e-3</td>
<td>7.2812</td>
</tr>
<tr>
<td>30</td>
<td>0.7611</td>
<td>4.3440e-5</td>
<td>1.8057</td>
<td>8.4573e-4</td>
<td>7.5123</td>
</tr>
<tr>
<td>40</td>
<td>0.7811</td>
<td>2.516e-5</td>
<td>1.7780</td>
<td>4.8819e-4</td>
<td>7.7093</td>
</tr>
<tr>
<td>50</td>
<td>0.7967</td>
<td>1.6764e-5</td>
<td>1.7536</td>
<td>3.1868e-4</td>
<td>7.8633</td>
</tr>
<tr>
<td>60</td>
<td>0.8091</td>
<td>1.1918e-5</td>
<td>1.7341</td>
<td>2.2477e-4</td>
<td>7.9862</td>
</tr>
<tr>
<td>70</td>
<td>0.8193</td>
<td>8.9195e-6</td>
<td>1.7195</td>
<td>1.6720e-4</td>
<td>8.0864</td>
</tr>
<tr>
<td>80</td>
<td>0.8278</td>
<td>6.9315e-6</td>
<td>1.7095</td>
<td>1.2934e-4</td>
<td>8.1703</td>
</tr>
<tr>
<td>90</td>
<td>0.8356</td>
<td>5.5486e-6</td>
<td>1.7036</td>
<td>1.0317e-4</td>
<td>8.2478</td>
</tr>
<tr>
<td>100</td>
<td>0.8418</td>
<td>4.5390e-6</td>
<td>1.7006</td>
<td>8.4185e-5</td>
<td>8.3087</td>
</tr>
</tbody>
</table>

The constant $C$ in the table is taken as $C = 1/\pi^2$.

Next, we present some examples of the a posteriori error for the following bi-harmonic problem.

\[
\Delta^2 u = f \quad \text{in } \Omega, \quad u = \partial_n u = 0 \quad \text{on } \partial\Omega, \quad (3.3)
\]

where $\Omega = (0,1)^2$ and

\[
f \equiv f(x,y) = 8\left(3x^2(1-x)^2 + xy(x-1)(y-1)(2x-1)(2y-1) + 3y^2(1-y)^2\right).
\]

The exact solution of (3.3) is given by $u \equiv u(x,y) = x^2y^2(1-x)^2(1-y)^2$.

Table 2 shows numerical results for the a priori and a posteriori error estimates in (3.1) and (3.2), respectively. Note that $\|f\|_{L^2} = \sqrt{992/175} < 2.3809$.

Table 2: Numerical results for the a priori and a posteriori estimates in (3.1) and (3.2)

<table>
<thead>
<tr>
<th>1/h</th>
<th>(3.1)</th>
<th>(3.2)</th>
<th>$|\Delta u - \Delta u_h|_{L^2}$</th>
<th>$|\Delta u_h - \overline{\Delta u_h}|_{L^2}$</th>
<th>$|f - \Delta u_h|_{L^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.7201e-2</td>
<td>7.8301e-3</td>
<td>5.0527e-4</td>
<td>4.4700e-4</td>
<td>1.2798</td>
</tr>
<tr>
<td>20</td>
<td>4.3909e-3</td>
<td>1.9680e-3</td>
<td>1.2595e-4</td>
<td>1.1897e-4</td>
<td>0.9047</td>
</tr>
<tr>
<td>30</td>
<td>2.0134e-3</td>
<td>8.7011e-4</td>
<td>5.5883e-5</td>
<td>5.3843e-5</td>
<td>0.7385</td>
</tr>
<tr>
<td>40</td>
<td>1.1623e-3</td>
<td>4.8659e-4</td>
<td>3.1376e-5</td>
<td>3.0521e-5</td>
<td>0.6394</td>
</tr>
<tr>
<td>50</td>
<td>7.5873e-4</td>
<td>3.0962e-4</td>
<td>2.0034e-5</td>
<td>1.9598e-5</td>
<td>0.5716</td>
</tr>
<tr>
<td>60</td>
<td>5.3510e-4</td>
<td>2.1363e-4</td>
<td>1.3863e-5</td>
<td>1.3611e-5</td>
<td>0.5215</td>
</tr>
<tr>
<td>70</td>
<td>3.9809e-4</td>
<td>1.5609e-4</td>
<td>1.0154e-5</td>
<td>9.9966e-6</td>
<td>0.4826</td>
</tr>
<tr>
<td>80</td>
<td>3.0795e-4</td>
<td>1.1870e-4</td>
<td>7.7390e-6</td>
<td>7.6331e-6</td>
<td>0.4511</td>
</tr>
<tr>
<td>90</td>
<td>2.4561e-4</td>
<td>9.3310e-5</td>
<td>6.0946e-6</td>
<td>6.0203e-6</td>
<td>0.4250</td>
</tr>
<tr>
<td>100</td>
<td>2.0042e-4</td>
<td>7.5133e-5</td>
<td>4.9152e-6</td>
<td>4.8611e-6</td>
<td>0.4029</td>
</tr>
</tbody>
</table>
All the computations were carried out by MATLAB on the Dell Precision 650 Workstation (Intel Xeon Dual CPU 3.20GHz).

References


