A Numerical Verification Method for Solutions of Singularly Perturbed Problems with Nonlinearity

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Abstract

In order to verify the solutions of nonlinear boundary value problems by Nakao's computer-assisted numerical method, it is required to find a constant, as sharp as possible, in the a priori error estimates for the finite element approximation of some simple linear problems. For singularly perturbed problems, however, generally it is known that the perturbation term produces a bad effect on the a priori error estimates, i.e., leads to a large constant, if we use the usual approximation methods. In this paper, we propose some verification algorithms for solutions of singularly perturbed problems with nonlinearity by using the constant obtained in the a priori error estimates based on the exponential fitting method with Green's function. Some numerical examples which confirm us the effectiveness of our method are presented.

Key word numerical verification, singularly perturbed problem, finite element method, a priori constant.

1 Introduction

The numerical verification method for solutions of nonlinear differential equations realizes a mathematically rigorous analysis on computer, which is often effectively applied to the problem for which usual theoretical approaches no longer work. However, there have been no practical verification methods which are suitable to the singularly perturbed problem up to the present. In this paper, we consider the verification method of solutions for the following singularly perturbed problem with nonlinearity including parameter ε ($1 \gg \varepsilon > 0$):

$$\left\{ \begin{array}{rcl} Lu \ \equiv \ -\varepsilon u'' - b(x)u' + c(x)u & = & f(u) \quad \text{in} \quad (0,1), \\ u(0) = u(1) & = & 0, \end{array} \right. \tag{1.1}$$

where $f : L^\infty(0,1) \to L^\infty(0,1)$ is a bounded continuous map, and we suppose that $b(x), c(x) \in W^1_\infty(0,1)$, and that there exists a constant γ such that $c(x) \geq \gamma > 0$.

Our arguments below are based on the finite element method on the interval $(0,1)$ with mesh : $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$. Let h and $h_{\min}$ denote the maximum and the minimum width of subintervals, respectively. We denote the trial and the test spaces by S_h and $V_h \subset L^\infty(0,1) \cap H^1_0(0,1)$, respectively. And we define the piecewise constant functions $\bar{b} = \bar{b}(x)$, $\bar{c} = \bar{c}(x) \in L^\infty(0,1)$ by

$$\bar{b}(x) = \frac{b(x_{i-1}) + b(x_i)}{2}, \quad \bar{c}(x) = \frac{c(x_{i-1}) + c(x_i)}{2}, \qquad x \in (x_{i-1}, x_i),$$

for $i = 1, \cdots, n$.

Here, we try to extend and apply Nakao's verification method [2] for the application to the singularly perturbed problem (1.1). In this method, it plays an important and essential role to get an a priori error estimation of the finite element solution of the linear problem, usually based on H^1_0-projection. For the singularly perturbed problems, however, when the usual finite element schemes are applied, it is known that the constant in the a priori error estimates increases as the perturbation parameter ε decreases. That is, from the error estimates of the H^1_0-projection to the solution of $-\varepsilon\phi'' = g$ ($g \in L^\infty(0,1)$), it is known that, if we use piecewise polynomials then we have

$$\|\phi - P_{H^1_0}\phi\|_\infty \leq C(\varepsilon)h^2\|g\|_\infty, \tag{1.2}$$

where $C(\varepsilon) \to \infty$ if $\varepsilon \to 0$. The estimates (1.2) imply that using such an approximate method would lead the inefficient computational procedure for the verification for small ε. Therefore, in this paper we paid attention to the exponential fitting technique, $\bar{L}$-spline method, studied by M. Stynes and E. O'Riordan [4][6][7][8]. That is, we carefully and numerically estimated various kinds of constants appearing in the error analyses of their methods. And we succeed in estimating an a priori constant, such as in (1.2), numerically with independent of or less dependent on the perturbation parameter, and to apply it to the verification for solutions of the singularly perturbed problem with nonlinearity (1.1).

In Section 2, we show the computational result of the constructive a priori error estimation of constants to the linear problem. Especially, as for the convection-diffusion problem of the subsection 2.1, we obtained the error estimation of $O(h)$ which is independent of ε. In the subsection 2.2, the theoretical analysis yields some constructive error estimations to the reaction-diffusion problem, which can be more effectively used for the verification of the nonlinear problem compared with usual methods, even though it still depends on the perturbation parameter. In the section 3, an actual verification algorithm is shown, and some verification results are presented in the section 4.

2 Constructive a priori error estimates for a linear singularly perturbed problem using Green's function

In this section, we try to compute the constant in the a priori error estimation of finite element approximations for linear convection and reaction diffusion equations. We determine two different kinds of constants corresponding to the convection and the reaction diffusion problems, respectively.

2.1 Convection Diffusion Problem

We first consider the following linear convection diffusion problems.

$$\left\{ \begin{array}{rcl} L\phi \;\equiv\; -\varepsilon\phi'' - b(x)\phi' + c(x)\phi & = & g \quad \text{in} \quad (0,1), \\ \phi(0) = \phi(1) & = & 0, \end{array} \right. \tag{2.1}$$

where $g \in L^\infty(0,1)$, and $b(x), c(x) \in W^1_\infty(0,1)$ are given functions satisfying $b(x) \geq \beta > 0$ and $c(x) \geq \gamma > 0$.

Now we define the bilinear form of (2.1) by, for each $\varphi, \psi \in H^1_0(0,1)$,

$$\begin{aligned} a(\varphi,\psi) &\equiv \varepsilon(\varphi',\psi') - (b\varphi',\psi) + (c\varphi,\psi), \\ a_h(\varphi,\psi) &\equiv \varepsilon(\varphi',\psi') - (\bar{b}\varphi',\psi) + (\bar{c}\varphi,\psi), \end{aligned}$$

where $(\cdot,\cdot)$ denotes L^2 inner product on $(0,1)$. Then, the projection $P_h : H^1_0 \to S_h$ is defined as

$$a(\phi - P_h\phi, \psi_h) \;=\; 0, \quad \text{for all } \psi_h \in V_h. \tag{2.2}$$

And we also define the approximation $P^\varepsilon_h\phi \equiv \phi^\varepsilon_h \in S_h$ of solution ϕ to (2.1), which we call the P^ε_h-projection, as follows :

$$a_h(\phi^\varepsilon_h, \psi_h) \;=\; a(\phi, \psi_h), \quad \text{for all } \psi_h \in V_h. \tag{2.3}$$

Now, M. Stynes and E. O'Riordan [7] introduced the following $\bar{L}$-spline $\{\varphi_i\}^n_{i=1}$ and $\bar{L}^*$-spline $\{\psi_i\}^n_{i=1}$ which constitute bases of S_h and V_h, respectively, satisfying for $i = 1, \cdots, n$

$$\begin{array}{rcrcl} \bar{L}\varphi_i & \equiv & -\varepsilon\varphi_i'' - \bar{b}\varphi_i' + \bar{c}\varphi_i & = & 0 \quad \text{in} \quad [0,1]\backslash\{x_1, \cdots x_n\}, \\ & & \varphi_i(x_k) & = & \delta^k_i \quad \text{for} \quad k = 0, \cdots n+1, \end{array}$$

$$\begin{array}{rcrcl} \bar{L}^*\psi_i & \equiv & -\varepsilon\psi_i'' + \bar{b}\psi_i' + \bar{c}\psi_i & = & 0 \quad \text{in} \quad [0,1]\backslash\{x_1, \cdots x_n\}, \\ & & \psi_i(x_k) & = & \delta^k_i \quad \text{for} \quad k = 0, \cdots n+1, \end{array}$$

where δ^k_i stands for Kronecker's delta.

We now define Green's function $G_i = G(x, x_i)$ by the following equation, which is spanned by $\{\psi_i\}^n_{i=1}$:

$$a_h(w, G_i) = w(x_i) \qquad \text{for all } w \in H^1_0(0,1).$$

Remark 1 *[7] For each $i \in \{1, \cdots, n\}$, Green's function $G_i(\cdot) \in C[0,1]$ is characterized by*

$$-\varepsilon G_i''(x) + \bar{b}G_i'(x) + \bar{c}G_i(x) = 0 \quad \text{in} \quad [0,1]\backslash\{x_1, \cdots, x_n\},$$

$$G_i(0) = G_i(1) = 0,$$

$$\lim_{x \to x_k^+} (\varepsilon G_i'(x) - \bar{b}G_i(x)) - \lim_{x \to x_k^-} (\varepsilon G_i'(x) - \bar{b}G_i(x)) = -\delta_i^k,$$

where $x_k^- \equiv x_k - 0$ and $x_k^+ \equiv x_k + 0$. Moreover, $G_i(x)$ lies in V_h.

Lemma 2.1 *For each $i \in \{1, \cdots, n\}$, Green's function $G_i(\cdot)$ satisfies*

$$\varepsilon G_i'(x) - \bar{b}G_i(x) = \varepsilon G_i'(0) + \int_0^x \bar{c}G_i(t)\,dt - H_i(x), \quad x \in [0,1]\backslash\{x_1, \cdots x_n\},$$

where

$$H_i(x) = \begin{cases} 1 & \text{if} \quad x \geq x_i, \\ 0 & \text{if} \quad x < x_i. \end{cases}$$

Proof: By Remark 1, we have

$$\begin{aligned} \int_0^x \bar{c}G_i(t)\,dt &= \sum_{k=1}^{j-1} \int_{x_{k-1}}^{x_k} \big(\varepsilon G_i''(t) - \bar{b}G_i'(t)\big)\,dt + \int_{x_{j-1}}^{x} \big(\varepsilon G_i''(t) - \bar{b}G_i'(t)\big)\,dt \\ &= \sum_{k=1}^{j-1} [\varepsilon G_i'(t) - \bar{b}G_i(t)]_{x_{k-1}}^{x_k} + [\varepsilon G_i'(t) - \bar{b}G_i(t)]_{x_{j-1}}^{x} \\ &= \big(\varepsilon G_i'(x) - \bar{b}G_i(x)\big) - \varepsilon G_i'(0) \\ &\quad + \sum_{k=1}^{j-1} \left[\big(\varepsilon G_i'(x_k^-) - \bar{b}G_i(x_k^-)\big) - \big(\varepsilon G_i'(x_k^+) - \bar{b}G_i(x_k^+)\big)\right], \end{aligned}$$

which proves the lemma. ∎

Lemma 2.2 *For each $i \in \{1, \cdots, n\}$ and for arbitrary $x \in (0,1)$, we have*

$$0 \leq G_i(x) \leq \frac{1}{\beta}.$$

Proof: From [7], it follows that $G_i(x) \geq 0$ for all $i \in \{1, \cdots, n\}$ (cf. the proof of Lemma 2.4 in this paper). We assume that there exists $z \in [x_{j-1}, x_j]$ such that $G_i(z) > 1/\beta$. Then, from Lemma 2.1 we have

$$\begin{aligned} \varepsilon G_i'(z) &= \bar{b}G_i(z) + \varepsilon G_i'(0) + \int_0^z \bar{c}G_i(t)\,dt - H_i(z) \\ &\geq \bar{b}G_i(z) - 1 \\ &\geq \beta G_i(z) - 1 \\ &> 0, \end{aligned}$$

where we have used the fact that $G_i'(0) \geq 0$ because $G_i(x) \geq 0$ and $G_i(0) = 0$. Hence we obtain $1/\beta < G_i(z) < G_i(x_j)$. It means that $1/\beta < G_i(1)$ from the inductive argument, which contradicts with $G_i(1) = 0$. ∎

Lemma 2.3 *Let ϕ be a solution of (2.1). Then we have that $\|\phi\|_\infty \leq K\|g\|_\infty$, where $K = 1/\max\{\beta, \gamma\}$ and for all $x \in (0,1)$, it follows*

$$|\phi'(x)| \leq \left(\frac{1}{\varepsilon} C_1 \mathrm{e}^{-\frac{\beta}{\varepsilon}x} + C_2 \mathrm{e}^{-\frac{\beta}{\varepsilon}x} + C_3\right) \|g\|_\infty,$$

where

$$C_1 = \frac{2}{\beta} K \|b\|_\infty^2, \quad C_2 = \frac{1}{\beta}(1 + K\|c\|_\infty + 2K\|b'\|), \quad C_3 = \frac{1}{\beta}(1 + K\|c\|_\infty).$$

Proof: We first show that $\|\phi\|_\infty \leq 1/\max\{\beta,\gamma\}\|g\|_\infty$. Let $y_1(x) = K_1(1 - x)\|g\|_\infty \pm \phi(x)$ and $y_2(x) = K_2\|g\|_\infty \pm \phi(x)$ where K_1, K_2 are positive constants. Then we have that $y_1(0) \geq y_1(1) = 0$ and $y_2(0) = y_2(1) \geq 0$. Hence if $K_1 = 1/\beta$ and $K_2 = 1/\gamma$ then we find

$$\begin{aligned} Ly_1 &= K_1\left[b(x) + (1-x)c(x)\right]\|g\|_\infty \pm g \\ &\geq K_1\beta\|g\|_\infty \pm g \\ &\geq 0, \end{aligned} \qquad \begin{aligned} Ly_2 &= K_2 c(x)\|g\|_\infty \pm g \\ &\geq K_2\gamma\|g\|_\infty \pm g \\ &\geq 0. \end{aligned}$$

Therefore, we obtain $\|\phi\|_\infty \leq K\|g\|_\infty$ by the maximum principle.

Now let $g_0(x) \equiv g(x) - c(x)\phi$, and let

$$g_1(x) \equiv \int_0^x g_0(t)\mathrm{e}^{\frac{1}{\varepsilon}\hat{b}(t)}\,dt, \qquad g_2(x) \equiv \int_x^1 g_0(t)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)}\,dt,$$

where $\hat{b}(x) := \displaystyle\int_0^x b(t)\,dt > 0$. Then we get

$$(-\varepsilon\phi'\mathrm{e}^{\frac{1}{\varepsilon}\hat{b}(x)})' = (-\varepsilon\phi'' - b(x)\phi')\mathrm{e}^{\frac{1}{\varepsilon}\hat{b}(x)}, \tag{2.4}$$

$$(-\varepsilon\phi'\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(x)})' = (-\varepsilon\phi'' + b(x)\phi')\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(x)}. \tag{2.5}$$

From (2.4)-(2.5) we have

$$\int_0^x (-\varepsilon\phi'\mathrm{e}^{\frac{1}{\varepsilon}\hat{b}(t)})'\,dt = -\varepsilon\phi'(x)\mathrm{e}^{\frac{1}{\varepsilon}\hat{b}(x)} + \varepsilon\phi'(0)\mathrm{e}^{\frac{1}{\varepsilon}\hat{b}(0)} = g_1(x), \tag{2.6}$$

$$\begin{aligned} \int_x^1 (-\varepsilon\phi'\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)})'\,dt &= -\varepsilon\phi'(1)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)} + \varepsilon\phi'(x)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(x)} \\ &= g_2(x) + 2\int_x^1 b(t)\phi'(t)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)}\,dt \\ &= g_2(x) + 2\left[b(t)\phi(t)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)}\right]_x^1 - 2\kappa(x), \end{aligned} \tag{2.7}$$

where

$$\kappa(x) \equiv \int_x^1 \phi(t) \left(b'(t) \mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)} - \frac{b(t)^2}{\varepsilon} \mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)} \right) dt.$$

Rewriting (2.6) as

$$-\varepsilon\phi'(x) + \varepsilon\phi'(0)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(x)} = g_1(x)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(x)}, \tag{2.8}$$

we have the following linear system from (2.7) and (2.8).

$$\begin{aligned} \varepsilon\phi'(1) - \varepsilon\phi'(0)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)} &= -g_1(1)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)}, \\ -\varepsilon\phi'(1)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)} + \varepsilon\phi'(0) &= g_2(0) - 2\kappa(0). \end{aligned}$$

Since $\hat{b}(1) := \displaystyle\int_0^1 b(t)\, dt \geq \beta > 0$, this system is nonsingular so that

$$\begin{pmatrix} 1 & -\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)} \\ -\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)} & 1 \end{pmatrix}^{-1} = \frac{1}{1-\mathrm{e}^{-\frac{2}{\varepsilon}\hat{b}(1)}} \begin{pmatrix} 1 & \mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)} \\ \mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)} & 1 \end{pmatrix} \geq 0.$$

Therefore we obtain

$$\begin{aligned} \begin{pmatrix} |\phi'(1)| \\ |\phi'(0)| \end{pmatrix} &\leq \frac{1}{\varepsilon}\frac{1}{1-\mathrm{e}^{-\frac{2}{\varepsilon}\hat{b}(1)}} \begin{pmatrix} 1 & \mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)} \\ \mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)} & 1 \end{pmatrix} \begin{pmatrix} |g_1(1)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)}| \\ |g_2(0)| + 2|\kappa(0)| \end{pmatrix} \\ &\leq \frac{1}{\varepsilon}\frac{1}{1-\mathrm{e}^{-\frac{2}{\varepsilon}\beta}} \begin{pmatrix} 1 & \mathrm{e}^{-\frac{\beta}{\varepsilon}} \\ \mathrm{e}^{-\frac{\beta}{\varepsilon}} & 1 \end{pmatrix} \begin{pmatrix} |g_1(1)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)}| \\ |g_2(0)| + 2|\kappa(0)| \end{pmatrix}. \end{aligned}$$

And we have

$$\begin{aligned} \left| g_1(1)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(1)} \right| = \left| \int_0^1 g_0(t)\mathrm{e}^{-\frac{1}{\varepsilon}[\hat{b}(1)-\hat{b}(t)]}\, dt \right| &\leq \|g_0\|_\infty \int_0^1 \mathrm{e}^{-\frac{1}{\varepsilon}[\hat{b}(1)-\hat{b}(t)]}\, dt \\ &\leq \|g_0\|_\infty \int_0^1 \mathrm{e}^{-\frac{\beta}{\varepsilon}(1-t)}\, dt \\ &= \frac{\varepsilon}{\beta}\|g_0\|_\infty \left(1 - \mathrm{e}^{-\frac{\beta}{\varepsilon}}\right), \end{aligned}$$

$$\begin{aligned} |g_2(0)| = \left| \int_0^1 g_0(t)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)}\, dt \right| &\leq \|g_0\|_\infty \int_0^1 \mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)}\, dt \\ &\leq \|g_0\|_\infty \int_0^1 \mathrm{e}^{-\frac{\beta}{\varepsilon}t}\, dt \\ &= \frac{\varepsilon}{\beta}\|g_0\|_\infty \left(1 - \mathrm{e}^{-\frac{\beta}{\varepsilon}}\right). \end{aligned}$$

Moreover, we have

$$
\begin{aligned}
|\kappa(0)| &= \left|\int_0^1 \phi(t)\left(b'(t)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)} - \frac{b(t)^2}{\varepsilon}\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)}\right)\,dt\right| \\
&\le \left|\int_0^1 b'(t)\phi(t)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)}\,dt\right| + \frac{1}{\varepsilon}\left|\int_0^1 b(t)^2\phi(t)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)}\,dt\right| \\
&\le \|b'\|_\infty\|\phi\|_\infty\int_0^1 \mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)}\,dt + \frac{1}{\varepsilon}\|b\|_\infty^2\|\phi\|_\infty\int_0^1 \mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(t)}\,dt \\
&\le \|b'\|_\infty\|\phi\|_\infty\int_0^1 \mathrm{e}^{-\frac{\beta}{\varepsilon}t}\,dt + \frac{1}{\varepsilon}\|b\|_\infty^2\|\phi\|_\infty\int_0^1 \mathrm{e}^{-\frac{\beta}{\varepsilon}t}\,dt \\
&= \|b'\|_\infty\|\phi\|_\infty\frac{\varepsilon}{\beta}(1-\mathrm{e}^{-\frac{\beta}{\varepsilon}}) + \|b\|_\infty^2\|\phi\|_\infty\frac{1}{\beta}(1-\mathrm{e}^{-\frac{\beta}{\varepsilon}}).
\end{aligned}
$$

Since $\|g_0\|_\infty \le (1+K\|c\|_\infty)\|g\|_\infty$, we obtain

$$
\begin{aligned}
|\phi'(0)| &\le \frac{1-\mathrm{e}^{-\frac{\beta}{\varepsilon}}}{1-\mathrm{e}^{-\frac{2\beta}{\varepsilon}}}\left[\frac{1+\mathrm{e}^{-\frac{\beta}{\varepsilon}}}{\beta}\|g_0\|_\infty + \frac{2}{\beta}\|b'\|_\infty\|\phi\|_\infty + \frac{2}{\varepsilon\beta}\|b\|_\infty^2\|\phi\|_\infty\right] \\
&\le \frac{1}{1+\mathrm{e}^{-\frac{\beta}{\varepsilon}}}\left[\frac{1+\mathrm{e}^{-\frac{\beta}{\varepsilon}}}{\beta}\|g_0\|_\infty + \frac{2}{\beta}\|b'\|_\infty\|\phi\|_\infty + \frac{2}{\varepsilon\beta}\|b\|_\infty^2\|\phi\|_\infty\right] \\
&\le \frac{1}{\beta}(1+K\|c\|_\infty + 2K\|b'\|)\|g\|_\infty + \frac{2}{\varepsilon\beta}K\|b\|_\infty^2\|g\|_\infty. \qquad (2.9)
\end{aligned}
$$

Finally, we have the following estimate

$$
\begin{aligned}
|g_1(x)\mathrm{e}^{-\frac{1}{\varepsilon}\hat{b}(x)}| = \left|\int_0^x g_0(t)\mathrm{e}^{-\frac{1}{\varepsilon}[\hat{b}(x)-\hat{b}(t)]}\,dt\right| &\le \|g_0\|_\infty\int_0^x \mathrm{e}^{-\frac{1}{\varepsilon}[\hat{b}(x)-\hat{b}(t)]}\,dt \\
&\le \|g_0\|_\infty\int_0^x \mathrm{e}^{-\frac{\beta}{\varepsilon}(x-t)}\,dt \\
&\le \frac{\varepsilon}{\beta}(1+K\|c\|_\infty)\|g\|_\infty. \qquad (2.10)
\end{aligned}
$$

Combining (2.9) and (2.10) with (2.8), we obtain the desired conclusion. ∎

Theorem 2.1 *Let ϕ and ϕ_h^ε be solutions of (2.1) and (2.3), respectively. Then*

$$|(\phi-\phi_h^\varepsilon)(x_i)| \le C_0 h\|g\|_\infty, \qquad i = 1,2,\cdots,n, \qquad (2.11)$$

where

$$C_0 = \frac{1}{\beta^2}\|b'\|_\infty\left(C_1 + \varepsilon C_2 + \beta C_3\right) + \frac{1}{\beta}K\|c'\|_\infty.$$

Here, constants C_i, $1 \le i \le 3$, and K are defined in Lemma 2.3.

Proof: From the property of Green's function, we have

$$
\begin{aligned}
(\phi-\phi_h^\varepsilon)(x_i) &= a_h(\phi-\phi_h^\varepsilon, G_i) = a_h(\phi, G_i) - (g, G_i) \\
&= -(\phi', (\bar{b}-b)G_i) + (\phi, (\bar{c}-c)G_i).
\end{aligned}
$$

Hence, we have

$$|(\phi - \phi_h^\varepsilon)(x_i)| \le \|\phi'\|_{L_1}\|\bar{b} - b\|_\infty \|G_i\|_\infty + \|\phi\|_\infty \|\bar{c} - c\|_{L_1}\|G_i\|_\infty.$$

Now we estimate each term in the above. First, note that

$$\begin{aligned}
\|\bar{b} - b\|_\infty &\le \max_i \int_{x_{i-1}}^{x_i} |b'(t)|\, dt \le \|b'\|_\infty h,\\
\|\bar{c} - c\|_{L_1} &\le \int_0^1 \|\bar{c} - c\|_\infty \, dt \le \|c'\|_\infty h.
\end{aligned}$$

Since the estimations of $\|\phi\|_\infty$ and $\|G_i\|_\infty$ follow from Lemma 2.2 and 2.3, respectively, it suffices to bound $\|\phi'\|_{L_1}$. From Lemma 2.3, we obtain

$$\begin{aligned}
\|\phi'\|_{L_1} &= \int_0^1 |\phi'(x)| dx\\
&\le \int_0^1 \left(\frac{1}{\varepsilon} C_1 \mathrm{e}^{-\frac{\beta}{\varepsilon}x} + C_2 \mathrm{e}^{-\frac{\beta}{\varepsilon}x} + C_3 \right) \|g\|_\infty \, dx\\
&\le \left(\frac{1}{\beta} C_1 + \frac{\varepsilon}{\beta} C_2 + C_3 \right) \|g\|_\infty.
\end{aligned}$$

Thus the conclusion follows immediately. ∎

Theorem 2.2 *Let ϕ and ϕ_h^ε be solutions of (2.1) and (2.3), respectively. Then*

$$\|\phi - \phi_h^\varepsilon\|_\infty \le Ch\|g\|_\infty,$$

where $C \equiv C(\lambda) = \tau_1 + (2-\lambda)\tau_2$ for $\lambda \in [0,1)$ and τ_1, τ_2 satisfies

$$\tau_1 = \frac{4}{\beta^2 + 4\varepsilon\gamma}\|b'\|_\infty (C_1 + \varepsilon C_2), \qquad \tau_2 = \begin{cases} \tau & \text{if } C_0 = 0, \\ \max\{\tau,\ C_0/(1-\lambda)\} & \text{otherwise,} \end{cases}$$

where

$$\tau \equiv \frac{1}{\beta + (1-\lambda)\gamma h}\left[1 + (\|b'\|_\infty C_3 + K\|c'\|_\infty)h\right].$$

Moreover if $C_0 = 0$ then we can choose that $\lambda \in [0,1]$.
(Constants C_i, $0 \le i \le 3$, and K are same as in Theorem 2.1.)

Proof: We prove the theorem using the maximum principle. First, we consider the case of $C_0 = 0$. Notice that, by the maximum principle, if the function $r(x) \ge 0$ satisfies $\bar{L}(\pm(\phi - \phi_h^\varepsilon)) \le \bar{L}r(x)$ for each $x \in (x_{i-1}, x_i)$, then we have $\pm(\phi - \phi_h^\varepsilon) \le r(x)$, for all $x \in [x_{i-1}, x_i]$, because $\phi(x_{i-1}) = \phi_h^\varepsilon(x_{i-1})$ and $\phi(x_i) = \phi_h^\varepsilon(x_i)$. From this property, we derive the maximum norm estimates of local error in each subinterval.
For each $x \in (x_{i-1}, x_i)$, we have

$$\begin{aligned}
\bar{L}(\pm(\phi - \phi_h^\varepsilon)) &= \bar{L}(\pm\phi) - \bar{L}(\pm\phi_h^\varepsilon)\\
&= \pm(-\varepsilon\phi'' - \bar{b}\phi' + \bar{c}\phi)\\
&= \pm\{g - (\bar{b} - b)\phi' + (\bar{c} - c)\phi\}\\
&\le \|g\|_\infty + \|\bar{b} - b\|_\infty |\phi'| + \|\bar{c} - c\|_\infty \|\phi\|_\infty\\
&\le \|g\|_\infty + \|b'\|_\infty |\phi'| h + \|c'\|_\infty \|\phi\|_\infty h.
\end{aligned}$$

Hence we get from Lemma 2.3

$$\begin{aligned}\bar{L}(\pm(\phi - \phi_h^\varepsilon)) &\leq \left(\frac{1}{\varepsilon}C_1 + C_2\right) \mathrm{e}^{-\frac{\beta}{\varepsilon}x}\|b'\|_\infty h\|g\|_\infty \\ &\qquad + \left[1 + (C_3\|b'\|_\infty + K\|c'\|_\infty)h\right]\|g\|_\infty.\end{aligned}$$

Now, for each $\lambda \in [0,1)$, define $r(x)$ by

$$r(x) \equiv \tau_1 \mathrm{e}^{-\frac{\beta}{2\varepsilon}x} h\|g\|_\infty + \tau_2 \left[h + \lambda x_{i-1} + (1-\lambda)x_i - x\right]\|g\|_\infty. \tag{2.12}$$

Then we have

$$\begin{aligned}\bar{L}r(x) &= \tau_1 \left(-\frac{\beta^2}{4\varepsilon} + \frac{\beta\bar{b}}{2\varepsilon} + \bar{c}\right) \mathrm{e}^{-\frac{\beta}{2\varepsilon}x} h\|g\|_\infty \\ &\qquad + \tau_2 \left[\bar{b} + \bar{c}\,(h + \lambda x_{i-1} + (1-\lambda)x_i - x)\right]\|g\|_\infty \\ &\geq \tau_1 \left(\frac{\beta^2}{4\varepsilon} + \gamma\right) \mathrm{e}^{-\frac{\beta}{2\varepsilon}x} h\|g\|_\infty + \tau_2\left[\beta + (1-\lambda)\gamma h\right]\|g\|_\infty.\end{aligned}$$

Hence if

$$\begin{aligned}\tau_1 &\geq \frac{4}{\beta^2 + 4\varepsilon\gamma}(C_1 + \varepsilon C_2)\|b'\|_\infty, \\ \tau_2 &\geq \frac{1}{\beta + (1-\lambda)\gamma h}\left[1 + (\|b'\|_\infty C_3 + K\|c'\|_\infty)h\right],\end{aligned}$$

then, noting that $\mathrm{e}^{-\frac{\beta}{\varepsilon}x} \leq \mathrm{e}^{-\frac{\beta}{2\varepsilon}x}$, we have $\bar{L}(\pm(\phi - \phi_h^\varepsilon)) \leq \bar{L}r(x)$ on $[x_{i-1}, x_i]$. In this case, we can choose that $\lambda \in [0,1]$ since $r(x) \geq 0$ if $\lambda = 1$.

Next, consider the case of $C_0 \neq 0$. In that case, we also define $r(x)$ by (2.12), then we obtain a positive lower bound of $r(x)$ satisfying $r(x) \geq \tau_2(1-\lambda)h\|g\|_\infty$. Hence the function $r(x)$ satisfies the condition of the maximum principle for the corresponding error estimation if $\tau_2(1-\lambda)h\|g\|_\infty \geq C_0 h\|g\|_\infty$. Thus, we obtain the following condition on τ_2.

$$\tau_2 \geq \frac{C_0}{1-\lambda}.$$

Therefore, the proof is completed since $|r(x)| \leq \left[\tau_1 + (2-\lambda)\tau_2\right] h\|g\|_\infty$ in $[x_{i-1}, x_i]$. ∎

Note that the a priori constant C in this theorem still includes the perturbation parameter ε, but one can readily see that it is essentially independent of this parameter. As a special case, we have the following corollary.

Corollary 1 *If $b(x) \equiv \beta$ and $c(x) \equiv \gamma$ are constant, then*

$$\|\phi - \phi_h^\varepsilon\|_\infty \leq \frac{h}{\beta}\|g\|_\infty. \tag{2.13}$$

Moreover if $\beta \geq \gamma h$ then the constant appeared in the right hand side of (2.13) gives the minimum value of $C(\lambda)$ in Theorem 2.2 with respect to all $\lambda \in [0,1]$.

Proof: In this case, we have $\|b'\|_\infty = \|c'\|_\infty = 0$ in Theorem 2.2, it implies that $C_0 = 0$. Hence, we find

$$C(\lambda) = \frac{2-\lambda}{\beta + (1-\lambda)\gamma h}, \quad C'(\lambda) = \frac{\gamma h - \beta}{[\beta + (1-\lambda)\gamma h]^2}.$$

Therefore $C(1) = 1/\beta$, and, moreover, $C(\lambda)$ is a monotone decreasing function, if $\beta \geq \gamma h$. Thus, we have the desired conclusion immediately. ∎

2.2 Reaction Diffusion Problem

We consider the following linear reaction diffusion problems.

$$\begin{cases} L\phi \;\equiv\; -\varepsilon\phi'' + c(x)\phi \;=\; g \quad \text{in} \quad (0,1), \\ \qquad\qquad \phi(0) = \phi(1) \;=\; 0, \end{cases} \tag{2.14}$$

where $g \in L^\infty(0,1)$ and $c(x) \in W^1_\infty(0,1)$ with $c(x) \geq \gamma > 0$.

For all $\varphi, \psi \in H^1_0(0,1)$, we define the bilinear form associated with (2.14) by

$$\begin{aligned} a(\varphi,\psi) &\equiv \varepsilon(\varphi',\psi') + (c\varphi,\psi), \\ a_h(\varphi,\psi) &\equiv \varepsilon(\varphi',\psi') + (\bar{c}\varphi,\psi). \end{aligned}$$

Then, the projection $P_h : H^1_0 \to S_h$ is defined as

$$a(\phi - P_h\phi, \psi_h) \;=\; 0, \quad \text{for all } \psi_h \in S_h. \tag{2.15}$$

And we also define the approximation $P^\varepsilon_h\phi \equiv \phi^\varepsilon_h \in S_h$ of solution ϕ to (2.14), which we call the P^ε_h-projection, as follows :

$$a_h(\phi^\varepsilon_h, \psi_h) \;=\; a(\phi,\psi_h), \quad \text{for all } \psi_h \in S_h. \tag{2.16}$$

Then we define the basis $\{\varphi_i\}_{i=1}^n$ of S_h, $\bar{L}$-spline, by the solutions of the problems for $i = 1, \cdots, n$

$$\begin{aligned} -\varepsilon\varphi_i'' + \bar{c}\varphi_i &= 0 \quad \text{in} \quad [0,1]\backslash\{x_1, \cdots x_n\}, \\ \varphi_i(x_k) &= \delta_i^k \quad \text{for} \quad k = 0, \cdots n+1. \end{aligned}$$

Remark 2 *Note that the linear operator in (2.14) also satisfies the maximum principle. (see [4] as $b(x) \equiv 0$)*

As previously, we define Green's function $G_i = G(x, x_i)$ which is spanned by $\{\varphi_i\}_{i=1}^n$ by the solution of following equation.

$$a_h(w, G_i) = w(x_i) \qquad \text{for all } w \in H^1_0(0,1). \tag{2.17}$$

We have the following equivalent formulation of G_i.

Lemma 2.4 *For each $i \in \{1, \cdots, n\}$, Green's function $G_i(\cdot) \in C[0,1]$ is characterized by*

$$-\varepsilon G_i''(x) + \bar{c} G_i(x) \;=\; 0 \quad \text{in} \quad [0,1] \backslash \{x_1, \cdots, x_n\}, \tag{2.18}$$

$$G_i(0) = G_i(1) = 0, \tag{2.19}$$

$$\lim_{x \to x_k^+} (\varepsilon G_i'(x)) - \lim_{x \to x_k^-} (\varepsilon G_i'(x)) = -\delta_i^k, \tag{2.20}$$

where the notation that x_k^- and x_k^+ are the same as the one which was defined in Remark 1. The above $G_i(x)$ is well-defined and lies in S_h with $G_i(x) \geq 0$. Moreover let $R = (R_{k,j})$ be a matrix with $R_{k,j} = a_h(\varphi_k, \varphi_j)$, $(1 \leq k, j \leq N)$ then $\|G_i\|_\infty \leq \|R^{-1}\|_{\ell_\infty}$ where $\| \cdot \|_{\ell_\infty}$ means matrix maximum norm.

Proof: For each $i \in \{1, \cdots, n\}$, we set

$$G_i = \sum_{j=1}^{n} \alpha_j^i \varphi_j.$$

Then (2.17) is equivalent to the following linear system.

$$\sum_{j=1}^{n} \alpha_j^i a_h(\varphi_k, \varphi_j) = \delta_i^k, \quad k = 1, \cdots, n. \tag{2.21}$$

Observing that

$$\begin{aligned} R_{k,j} = a_h(\varphi_k, \varphi_j) &= \varepsilon(\varphi_k', \varphi_j') + \bar{c}(\varphi_k, \varphi_j) \\ &= \sum_{l=1}^{n+1} \varepsilon[\varphi_k \varphi_j']_{x_{l-1}}^{x_l} + (\varphi_k, -\varepsilon \varphi_j'' + \bar{c} \varphi_j) \\ &= \sum_{l=1}^{n+1} \varepsilon[\varphi_k \varphi_j']_{x_{l-1}}^{x_l}, \end{aligned}$$

and noting that R is a tri-diagonal matrix from the property of base functions, we have

$$R_{k,k} \;=\; \varepsilon[\varphi_k \varphi_k']_{x_{k-1}}^{x_k} + \varepsilon[\varphi_k \varphi_k']_{x_k}^{x_{k+1}} = \varepsilon \varphi_k'(x_k^-) - \varepsilon \varphi_k'(x_k^+) \;>\; 0,$$

$$\begin{aligned} R_{k,k-1} &= \varepsilon[\varphi_k \varphi_{k-1}']_{x_{k-1}}^{x_k} = \varepsilon \varphi_{k-1}'(x_k^-) \;<\; 0, \\ R_{k,k+1} &= \varepsilon[\varphi_k \varphi_{k+1}']_{x_k}^{x_{k+1}} = -\varepsilon \varphi_{k+1}'(x_k^+) \;<\; 0. \end{aligned} \tag{2.22}$$

From

$$\varphi_k(x) = \begin{cases} \sinh\left(\sqrt{\frac{\bar{c}}{\varepsilon}}(x - x_{k-1})\right) / \sinh\left(\sqrt{\frac{\bar{c}}{\varepsilon}} h_k\right) & \text{if} \quad x \in [x_{k-1}, x_k] \\ \sinh\left(\sqrt{\frac{\bar{c}}{\varepsilon}}(x_{k+1} - x)\right) / \sinh\left(\sqrt{\frac{\bar{c}}{\varepsilon}} h_{k+1}\right) & \text{if} \quad x \in [x_k, x_{k+1}] \\ 0 & \text{otherwise} \end{cases} \tag{2.23}$$

we have $\varphi'_{k+1}(x_k^-) = \varphi'_{k-1}(x_k^+) = 0$. Hence (2.21) can be rewritten as

$$
\begin{aligned}
\delta_i^k &= \alpha_{k-1}^i R_{k,k-1} + \alpha_k^i R_{k,k} + \alpha_{k+1}^i R_{k,k+1} \\
&= \varepsilon\left(\alpha_{k-1}^i \varphi'_{k-1} + \alpha_k^i \varphi'_k\right)(x_k^-) - \varepsilon\left(\alpha_k^i \varphi'_k + \alpha_{k+1}^i \varphi'_{k+1}\right)(x_k^+) \\
&= \varepsilon\left(\alpha_{k-1}^i \varphi'_{k-1} + \alpha_k^i \varphi'_k + \alpha_{k+1}^i \varphi'_{k+1}\right)(x_k^-) - \varepsilon\left(\alpha_{k-1}^i \varphi'_{k-1} + \alpha_k^i \varphi'_k + \alpha_{k+1}^i \varphi'_{k+1}\right)(x_k^+) \\
&= \varepsilon G'_i(x_k^-) - \varepsilon G'_i(x_k^+).
\end{aligned}
$$

Therefore it follows that conditions (2.18)-(2.20) are equivalent to (2.17).

Next, we show that the coefficient matrix R is an M-matrix [1]. This can be easily proved as below by the fact that R is a symmetric and Z-matrix (all off-diagonal elements are nonpositive) from the definition of $a_h(\cdot,\cdot)$ and (2.22). Since

$$
R_{k-1,k} = -\varepsilon\varphi'_k(x_{k-1}^+), \qquad R_{k+1,k} = \varepsilon\varphi'_k(x_{k+1}^-),
$$

we obtain

$$
\begin{aligned}
R_{k-1,k} + R_{k,k} + R_{k+1,k} &= -\varepsilon\varphi'_k(x_{k-1}^+) + \varepsilon\varphi'_k(x_k^-) - \varepsilon\varphi'_k(x_k^+) + \varepsilon\varphi'_k(x_{k+1}^-) \\
&= \Big(\int_{x_k}^{x_{k+1}} + \int_{x_{k-1}}^{x_k}\Big)\varepsilon\varphi''_k(t)\,dt \\
&= \Big(\int_{x_k}^{x_{k+1}} + \int_{x_{k-1}}^{x_k}\Big)\bar{c}\varphi_k(t)\,dt \\
&\geq \Big(\int_{x_k}^{x_{k+1}} + \int_{x_{k-1}}^{x_k}\Big)\gamma\varphi_k(t)\,dt \\
&> 0.
\end{aligned}
$$

Hence we have that R is a strictly diagonally dominant matrix, which means R is an M-matrix. Thus, $G_i(x)$ is well-defined and lies in S_h with $G_i(x) \geq 0$, because M-matrix is nonsingular, all elements of its inverse matrix are nonnegative and the components of the right-hand side of (2.21) are nonnegative.

Next we show that $G_i(x)$ attains the maximum value at the end point on each subinterval.

Let

$$
r_k(x) = \begin{cases} \frac{1}{h_k}(x - x_{k-1}) - \varphi_k(x) & \text{if} \quad x \in [x_{k-1}, x_k] \\ \frac{1}{h_{k+1}}(x_{k+1} - x) - \varphi_k(x) & \text{if} \quad x \in [x_k, x_{k+1}] \\ 0 & \qquad \text{otherwise} \end{cases}
$$

for $k = 1, \cdots, n$. Then, since $r_k(x_{i-1}) = r_k(x_i) = r_k(x_{i+1}) = 0$ and $\bar{L}r_k(x) > 0$ in each subinterval, we obtain $r_k(x) \geq 0$ from the maximum principle. And each Green's function satisfies $G_i(x) = \alpha_k^i\varphi_k(x) + \alpha_{k+1}^i\varphi_{k+1}(x)$ on each subinterval. Therefore it follows that

$$
\begin{aligned}
|\alpha_k^i\varphi_k(x) + \alpha_{k+1}^i\varphi_{k+1}(x)| &\leq |\alpha_k^i|\varphi_k(x) + |\alpha_{k+1}^i|\varphi_{k+1}(x) \\
&\leq \max(|\alpha_k^i|, |\alpha_{k+1}^i|)\cdot(\varphi_k(x) + \varphi_{k+1}(x)) \\
&\leq \max(|\alpha_k^i|, |\alpha_{k+1}^i|),
\end{aligned}
$$

because $r_k(x) \geq 0$. Thus, we get the following estimate.

$$\|G_i\|_\infty = \max_k |(R^{-1})_{k,i}| \leq \max_k \max_j |(R^{-1})_{k,j}| \leq \max_k \sum_{j=1}^{n} |(R^{-1})_{k,j}| = \|R^{-1}\|_{\ell_\infty},$$

Therefore the proof is completed. ∎

Theorem 2.3 *Let ϕ and ϕ_h^ε be solutions of (2.14) and (2.16), respectively. Then*

$$|(\phi - \phi_h^\varepsilon)(x_i)| \leq C_0 h \|g\|_\infty, \qquad i = 1, 2, \cdots, n, \tag{2.24}$$

where $C_0 = 1/\gamma \|R^{-1}\|_{\ell_\infty} \|c'\|_\infty$, where R is the same matrix as described in Lemma 2.4.

Proof: From the property of Green's function, we have

$$\begin{aligned} (\phi - \phi_h^\varepsilon)(x_i) &= a_h(\phi - \phi_h^\varepsilon, G_i) \\ &= (\phi, (\bar{c} - c)G_i). \end{aligned}$$

Hence the nodal errors can be estimated as

$$\begin{aligned} |(\phi - \phi_h^\varepsilon)(x_i)| &\leq \|\phi\|_\infty \|\bar{c} - c\|_{L_1} \|G_i\|_\infty \\ &\leq \frac{1}{\gamma} \|R^{-1}\|_{\ell_\infty} \|c'\|_\infty h \|g\|_\infty. \end{aligned}$$

∎

For the numerical computation of $\|R^{-1}\|_{\ell_\infty}$ with guaranteed error bound, refer, e.g., [3].

Theorem 2.4 *Let ϕ and ϕ_h^ε be solutions of (2.14) and (2.16), respectively. Then*

$$\|\phi - \phi_h^\varepsilon\|_\infty \leq C(h, \varepsilon) \|g\|_\infty$$

where

$$C(h, \varepsilon) = \max \left\{ \frac{1}{\gamma} \left(1 + \frac{1}{\gamma} \|c'\|_\infty h \right) \delta(h, \varepsilon) + 2\mathrm{e}^{-\sqrt{\frac{\gamma}{\varepsilon}} \frac{h_{\min}}{2}} C_0 h, \ C_0 h \right\},$$

$$\delta(h, \varepsilon) \equiv \left(1 - \mathrm{e}^{-\sqrt{\frac{\gamma}{\varepsilon}} \frac{h}{2}} \right)^2 < 1.$$

Here, C_0 is same as in Theorem 2.3.

Proof: We use the maximum principle like the convection diffusion case. We now constitute a function $r(x)$ which satisfies with $\bar{L}(\pm(\phi - \phi_h^\varepsilon)) \leq \bar{L} r(x)$ and $\pm(\phi - \phi_h^\varepsilon)(x_{i-1}) \leq r(x_{i-1})$, $\pm(\phi - \phi_h^\varepsilon)(x_i) \leq r(x_i)$ in $[x_{i-1}, x_i]$ for $i = 1, \cdots, n$ as below.

We first estimate $\bar{L}(\pm(\phi - \phi_h^\varepsilon))$ as follows.

$$\begin{aligned}
\bar{L}(\pm(\phi - \phi_h^\varepsilon)) &= \bar{L}(\pm\phi) - \bar{L}(\pm\phi_h^\varepsilon) = \pm(-\varepsilon\phi'' + \bar{c}\phi) = \pm\{g + (\bar{c} - c)\phi\} \\
&\le \|g\|_\infty + \|\bar{c} - c\|_\infty \|\phi\|_\infty \\
&\le \|g\|_\infty + \|c'\|_\infty \|\phi\|_\infty h \\
&\le \left(1 + \frac{1}{\gamma}\|c'\|_\infty h\right) \|g\|_\infty, \quad (=: \mathbf{g})
\end{aligned}$$

where we have used the estimate $\|\phi\|_\infty \le 1/\gamma \|g\|_\infty$ which is following by the similar argument to that in the proof of Lemma 2.3. Here we solve the following ordinary differential equation with $\hat{C}_0 := C_0 h \|g\|_\infty$.

$$\begin{aligned}
-\varepsilon r''(x) + \bar{c} r(x) &= \mathbf{g} \quad \text{in} \quad (x_{i-1}, x_i), \\
r(x_{i-1}) = r(x_i) &= \hat{C}_0.
\end{aligned} \tag{2.25}$$

Then the solution of (2.25) is written as

$$r(x) = \frac{1}{1 + \mathrm{e}^{-\sqrt{\frac{\bar{c}}{\varepsilon}}h}} \left[\frac{\mathbf{g}}{\bar{c}} \left(1 - \mathrm{e}^{\sqrt{\frac{\bar{c}}{\varepsilon}}(x_{i-1}-x)} - \mathrm{e}^{\sqrt{\frac{\bar{c}}{\varepsilon}}(x-x_i)} + \mathrm{e}^{-\sqrt{\frac{\bar{c}}{\varepsilon}}h}\right) + \hat{C}_0 \left(\mathrm{e}^{\sqrt{\frac{\bar{c}}{\varepsilon}}(x_{i-1}-x)} + \mathrm{e}^{\sqrt{\frac{\bar{c}}{\varepsilon}}(x-x_i)}\right)\right].$$

Since $r'(x) = 0$ at $x = (x_{i-1} + x_i)/2$, we obtain

$$|r(x)| \le \frac{1}{1 + \mathrm{e}^{-\sqrt{\frac{\bar{c}}{\varepsilon}}h}} \max \left\{ \frac{\mathbf{g}}{\bar{c}} \left(1 - \mathrm{e}^{-\sqrt{\frac{\bar{c}}{\varepsilon}}\frac{h_i}{2}}\right)^2 + 2\mathrm{e}^{-\sqrt{\frac{\bar{c}}{\varepsilon}}\frac{h_i}{2}} \hat{C}_0,\ \hat{C}_0 \right\},$$

where $h_i = x_i - x_{i-1}$. Therefore, the maximum principle completes the proof. ∎

3 Numerical Verification Algorithm

The singularly perturbed problem (1.1) is transformed to the following fixed point equation:

$$u = F(u),$$

with the notation $F(u) = L^{-1} f(u)$, where F becomes a compact operator from $L^\infty(0,1) \cap H_0^1(0,1)$ to itself. We apply the verification method similar to that in [2].

Instead of the H_0^1-projection in [2], using the projection P_h defined by (2.2) and (2.15), we have the following decomposition of the fixed point equation $u = F(u)$.

$$\begin{aligned}
P_h u &= P_h F(u), \\
(I - P_h) u &= (I - P_h) F(u).
\end{aligned}$$

Let $\hat{u}_h^\varepsilon$ be an approximate solution of (1.1) which satisfies $a_h(\hat{u}_h^\varepsilon, \psi_h) = (f(\hat{u}_h^\varepsilon), \psi_h)$, for all $\psi_h \in V_h$, and let

$$N_h(u) \equiv P_h u - [I - F'(\hat{u}_h^\varepsilon)]_h^{-1} (P_h u - P_h F(u)). \tag{3.1}$$

Here $[I-F'(\hat{u}_h^\varepsilon)]_h^{-1}$ is an inverse operator of $P_h(I-F'(\hat{u}_h^\varepsilon))|_{S_h} : S_h \to S_h$ and $F'(\hat{u}_h^\varepsilon)$ is a Fréchet derivative of $F(u)$ at $\hat{u}_h^\varepsilon$. Then (1.1) is equivalent to the following fixed point equation.

$$u = Tu, \qquad Tu \equiv N_h(u) + (I - P_h)F(u).$$

Defining the candidate set, a set expected to include the desired solutions, as $U \equiv \hat{u}_h^\varepsilon + W_h + [[\alpha]]_\infty \subset L^\infty(0,1) \cap H_0^1(0,1)$, where

$$\begin{aligned} W_h &\equiv \left\{ w_h \in S_h : w_h = \sum_{i=1}^{n} W_i \varphi_i,\ W_i = [-w_i, w_i],\ w_i \ge 0 \right\}, \\ [[\alpha]]_\infty &\equiv \left\{ \hat{\alpha} \in L^\infty(0,1) \cap H_0^1(0,1) : \|\hat{\alpha}\|_\infty \le \alpha \right\}, \end{aligned}$$

if the condition

$$N_h(U) \subset U_h, \tag{3.2}$$

$$(I - P_h)F(U) \subset [[\alpha]]_\infty \tag{3.3}$$

holds, then, by Schauder's fixed point theorem, there exists a function $\hat{u} \in U$ such that $L\hat{u} = f(\hat{u})$.

In order to estimate the error $\|\phi - P_h\phi\|_\infty$, we use following triangular inequality

$$\|\phi - P_h\phi\|_\infty \le \|\phi - P_h^\varepsilon\phi\|_\infty + \|P_h^\varepsilon\phi - P_h\phi\|_\infty. \tag{3.4}$$

For the first term, we use the estimation already discussed in the previous section. For the second term, by the fact that $a(P_h\phi, \psi_h) = a_h(P_h^\varepsilon\phi, \psi_h)$, for all $\psi_h \in V_h$, it is essentially independent of ε, because of Lemma 2.3, Theorems 2.1 and 2.3. Actually, it is easily seen that, by using matrices $A = (A_{i,j}) = [a(\varphi_j, \psi_i)]$ and $A^h = (A_{i,j}^h) = [a_h(\varphi_j, \psi_i)]$, we have $\|P_h^\varepsilon\phi - P_h\phi\|_\infty \le \mathbf{C}h\|g\|_\infty$, where $\mathbf{C} := 3\|A^{-1} - (A^h)^{-1}\|_{\ell_\infty}$, which is followed by the properties of basis of S_h, V_h.

From the viewpoint of the effectiveness of computational cost, we usually use the residual form below instead of the original equation (1.1) [9].

That is, for the singularly perturbed problem (1.1), we define the solution $\bar{u}$ of the following *linear* singularly perturbed problem.

$$\begin{aligned} -\varepsilon\bar{u}'' - b(x)\bar{u}' + c(x)\bar{u} &= f(\hat{u}_h^\varepsilon) \quad \text{in} \quad (0,1), \\ \bar{u}(0) = \bar{u}(1) &= 0. \end{aligned}$$

Then, defining $v_0 \equiv \bar{u} - \hat{u}_h^\varepsilon$ we get the following estimates

$$\|v_0\|_\infty \le Ch\|f(\hat{u}_h^\varepsilon)\|_\infty$$

by the same constant C in Theorems 2.2 and 2.4, because $\hat{u}_h^\varepsilon$ coincides with P_h^ε-projection of $\bar{u}$. Thus the concerned problem is reduced to finding $w := u - \bar{u}$ satisfying

$$\begin{aligned} -\varepsilon w'' - b(x)w' + c(x)w &= f(w + v_0 + \hat{u}_h^\varepsilon) - f(\hat{u}_h^\varepsilon) \quad \text{in} \quad (0,1), \\ w(0) = w(1) &= 0. \end{aligned} \tag{3.5}$$

Then, since the approximate solution of (3.5) is taken as 0, the candidate set for the solution is usually taken of the form $W \equiv W_h + [[\alpha]]_\infty$.

4 Numerical Examples

We now present some numerical examples below, to show the effectiveness of the Theorems 2.2 and 2.4. We employed the residual form (3.5) in our procedures of verification.

We first consider the following example of the convection diffusion problem.

Example 4.1

$$\begin{aligned} L_1 u \equiv -\varepsilon u'' - bu' + cu &= 1 - u^3 \quad \text{in} \quad (0,1), \\ u(0) = u(1) &= 0, \end{aligned}$$

where $b = 1/(2\pi)^2$, $c = 1$.

We omit detailed and actual computational procedures for checking conditions (3.2) and (3.3) (see, e.g., [2][9] etc.). The approximate solutions are shown in Figure 1.

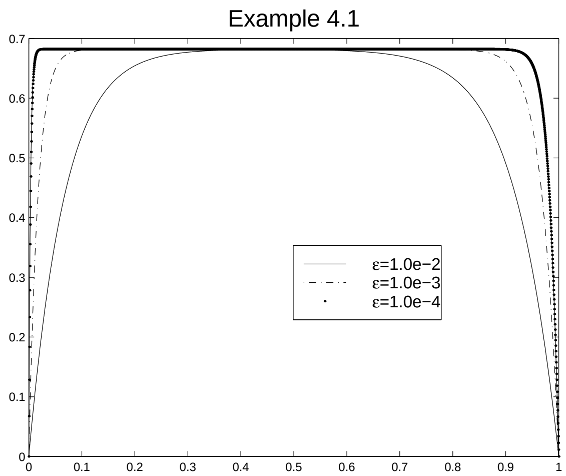

Figure 1: The approximate solutions of Example 4.1.

Table 1 shows the verification results for uniform mesh with $n = 999$ using $\bar{L}$-spline in this paper. We also illustrate, for comparison, in Tables 2 to 3 the results using the usual verification algorithm with piecewise linear functions as in [9]. We show that the distribution of mesh size for non-uniform mesh in Figure 3.

In Tables 1 to 3 and 4 to 6, the exact solution is enclosed by $\hat{u}_h^\varepsilon + W_h + [[\alpha]]_\infty + v_0$, i.e., "**Total**"s mean the total error of the approximate solution $\hat{u}_h^\varepsilon$. "**Fail**"s in tables mean that we could not get the solution u with guaranteed error bound, in the sense of infinite dimension, but we could get the approximate solution with guaranteed error bound, in the sense of finite dimension.

We next consider the following example of the reaction diffusion problem.

Example 4.2 (Allen-Cahn equation)

$$\begin{aligned} L_2 u \equiv -\varepsilon u'' + cu &= (c+1)u^2 - u^3 \quad \text{in} \quad (0,1), \\ & u(0) = u(1) = 0, \end{aligned}$$

Table 1: $\bar{L}$-spline (Uniform) for Example 4.1

$1/\varepsilon$	**Total**	α	$\|W_h\|_\infty$	$\|v_0\|_\infty$
100	4.8819e-2	2.7863e-3	1.9576e-2	3.7979e-2
1000	4.9337e-2	2.7858e-3	1.9554e-2	3.7979e-2
10000	5.1568e-2	2.7913e-3	1.9621e-2	3.7979e-2
100000	6.0336e-2	3.0415e-3	2.0178e-2	3.7979e-2
1000000	6.3029e-2	3.7266e-3	2.1801e-2	3.7979e-2

Table 2: Piecewise Linear (Uniform) for Example 4.1

$1/\varepsilon$	**Total**	α	$\|W_h\|_\infty$	$\|v_0\|_\infty$
100	9.4484e-3	1.0157e-5	1.1025e-4	9.3539e-3
1000	6.4661e-2	6.6864e-4	6.1947e-4	6.3552e-2
10000	9.1327e-1	8.7964e-2	3.8504e-3	8.2299e-1
100000	**Fail**	∞	∞	2.0192e+1
1000000	**Fail**	∞	∞	5.4777e+2

Table 3: Piecewise Linear (Non-Uniform) for Example 4.1

$1/\varepsilon$	**Total**	α	$\|W_h\|_\infty$	$\|v_0\|_\infty$
100	5.1665e-3	5.2703e-6	8.1393e-5	5.0948e-3
1000	2.4911e-2	1.4200e-4	2.9216e-4	2.4568e-2
10000	2.7563e-1	9.4780e-3	1.2795e-3	2.6568e-1
100000	**Fail**	∞	∞	6.7784e-0
1000000	**Fail**	∞	∞	2.0396e+2

where $c = 1/10$.

The approximate solutions are shown in Figure 2.

Tables 4 to 6 show the comparison of the effectiveness for $\bar{L}$-spline with uniform mesh and the usual piecewise linear finite element method with both uniform and nonuniform meshes for $n = 999$. Figure 3 shows that the distribution of mesh size for non-uniform mesh. By these tables, it is seen that if we use the uniformed mesh then $\bar{L}$-spline yields always better approximation than the usual piecewise linear finite element. "**Singular**" in Table 5 means that we could not get the approximate solution with guaranteed error bound, in the sense of finite dimension. As shown in Figure 4, the usefulness of $\bar{L}$-spline method should be more and more clear compared with the usual method when ε tends to be very small.

The numerical computations were carried out on a Dell Precision 650 Workstation by using INTLAB, a tool box in MATLAB developed by Rump [5] for self-validating algorithms.

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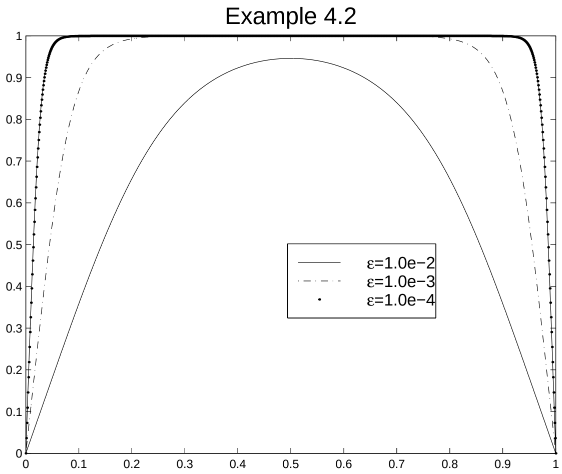

Figure 2: The approximate solutions of Example 4.2.

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Table 4: $\bar{L}$-spline (Uniform) for Example 4.2

$1/\varepsilon$	Total	α	$\|w_h\|_\infty$	$\|v_0\|_\infty$
1000	4.1860e-5	2.8959e-9	1.7250e-5	2.4666e-5
10000	4.3532e-4	2.9477e-7	1.8943e-4	2.4645e-4
100000	4.9995e-3	3.2202e-5	2.5339e-3	2.4621e-3
400000	2.7049e-2	7.1905e-4	1.6925e-2	9.8181e-3
500000	3.9078e-2	1.3660e-3	2.6007e-2	1.2256e-2
600000	5.8141e-2	2.5107e-3	4.1667e-2	1.4691e-2

Table 5: Piecewise Linear (Uniform) for Example 4.2

$1/\varepsilon$	Total	α	$\|w_h\|_\infty$	$\|v_0\|_\infty$
1000	4.1922e-5	2.4793e-9	2.6180e-5	1.5934e-5
10000	4.6950e-4	2.8032e-7	3.0774e-4	1.6279e-4
100000	6.7719e-3	4.4062e-5	5.0283e-3	1.7376e-3
400000	**Singular**	—	—	—
500000	**Fail**	∞	∞	9.8976e-3
600000	**Fail**	∞	∞	1.2015e-2

Table 6: Piecewise Linear (Non-Uniform) for Example 4.2

$1/\varepsilon$	Total	α	$\|w_h\|_\infty$	$\|v_0\|_\infty$
1000	3.0688e-5	7.3430e-9	1.8028e-5	1.3748e-5
10000	5.5890e-5	4.3732e-9	3.5027e-5	2.1099e-5
100000	6.3696e-4	5.0772e-7	4.2204e-4	2.1622e-4
400000	3.0715e-3	1.0212e-5	2.1771e-3	8.9695e-4
500000	**Fail**	∞	∞	1.1304e-3
600000	**Fail**	∞	∞	1.3611e-3

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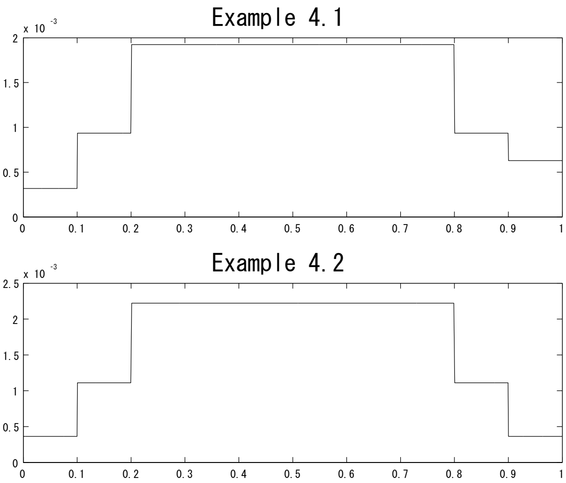

Figure 3: The distribution of mesh sizes.

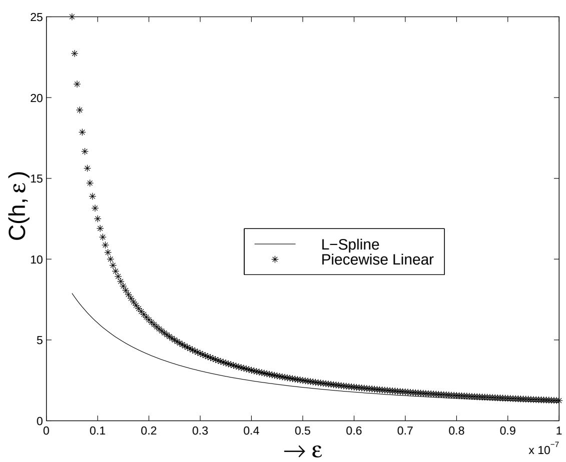

Figure 4: Constructive a priori constant $C(h, \varepsilon)$ for Example 4.2