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A COMPUTATIONAL APPROACH TO CONSTRUCTIVE A PRIORI AND A POSTERIORI ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS OF BI-HARMONIC PROBLEMS

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Abstract. In the numerical verification method of solutions for nonlinear fourth order elliptic equations, it is important to find a constant in the constructive a priori and a posteriori error estimates for the finite element approximation of bi-harmonic problems. We show these procedures by verified computational techniques using the Hermite spline functions for two dimensional rectangular domain. Several numerical examples which confirm the actual effectiveness of the method are presented.

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1 Introduction

In this paper, we consider the guaranteed error bounds of the finite element approximations for the following equation:

$$\begin{array}{rcccll} \Delta^2 u & = & f & & \text{in} & \Omega, \\ u & = & \partial_n u & = & 0 \text{ on} & \partial\Omega, \end{array} \tag{1.1}$$

where $f \in L^2(\Omega)$ with a rectangular domain Ω in R^2, and $\partial_n u$ denotes the outer normal derivative of u.

1.1 Notations

In the below, setting $\Omega := (0,1)^2$ for simplicity, we denote the L^2 inner product on Ω by $(\cdot,\cdot)_{L^2}$ and the norm by $\|\cdot\|_{L^2}$. We also denote the usual k-th order L^2 Sobolev space on Ω by $H^k(\Omega)$ for any positive integer k as well as the space $H_0^2(\Omega)$ by

$$H_0^2(\Omega) := \left\{ v \in H^2(\Omega) : \ v = \partial_n v = 0 \text{ on } \partial\Omega \right\}.$$

Let $S_h \subset H_0^2(\Omega)$, $n \equiv \dim S_h$, be a finite element space, under some rectangular mesh, which is spanned by two dimensional Hermite spline functions $\{\phi_i\}_{1\le i\le n}$ with homogeneous boundary conditions in H_0^2 sense [4]. Moreover, we define the space S_h^*, $n^* \equiv \dim S_h^*$, which is spanned by the basis $\{\phi_i^*\}_{1\le i\le n^*}$, as the finite element subspace of $H^2(\Omega)$, not of $H_0^2(\Omega)$, satisfying $S_h \subset S_h^*$ but $S_h \not\equiv S_h^* (n < n^*)$. Namely, the set of base functions $\{\phi_i^*\}_{1\le i\le n^*}$ consists of the elements in $\{\phi_i\}_{1\le i\le n}$ and the Hermite spline functions corresponding to the boundary nodes(*cf.* [5]).

Next, we define the H_0^2-projection $P_h : H_0^2(\Omega) \to S_h$ of $v \in H_0^2(\Omega)$ by

$$(\Delta v - \Delta P_h v, \Delta \phi_h)_{L^2} = 0, \quad \forall \phi_h \in S_h.$$

Moreover, we also define the L^2-projection $P_0 : L^2(\Omega) \to S_h^*$ of $v \in L^2(\Omega)$ by

$$(v - P_0 v, \phi_h^*)_{L^2} = 0, \quad \forall \phi_h^* \in S_h^*.$$

Let S_h^x and S_h^y denote the set of one dimensional Hermite spline functions on $(0,1)$ with homogeneous H_0^2 boundary conditions in $x-$ and $y-$ directions, respectively. Then, S_h is represented as the tensor product $S_h^x \otimes S_h^y$. Similarly, we have $S_h^* = S_h^{x^*} \otimes S_h^{y^*}$, where $S_h^{x^*}$, $S_h^{y^*}$ are spaces of one dimensional spline functions without boundary functions. In what follows, a parameter h stands for the maximum mesh size of the partition of the interval $(0,1)$.

In the x-direction, for $w \in H_0^2(0,1)$ and $w \in L_2(0,1)$, we also define the projections $P_2^x : H_0^2(0,1) \to S_h^x$ and $P_0^x : L_2(0,1) \to S_h^{x^*}$ by

$$(D_x^2 w - D_x^2 P_2^x w, D_x^2 \varphi_h)_{L^2} = 0, \quad \forall \varphi_h \in S_h^x,$$

and

$$(w - P_0^x w, \varphi_h^*)_{L^2} = 0, \quad \forall \varphi_h^* \in S_h^{x^*},$$

respectively. For P_2^y and P_0^y, analogously defined in the y-direction.

1.2 Motivation

Let $u_h \in S_h$ be an approximate solution of (1.1) satisfying

$$(\Delta u_h, \Delta \phi_h)_{L^2} = (f, \phi_h)_{L^2} \quad \forall \phi_h \in S_h.$$

Then, note that we have $u_h = P_h u$ by the definition and that the solution u of (1.1) belongs to $H_0^2(\Omega) \cap H^4(\Omega)$([1]). Therefore, in what follows, we will discuss on the error estimates for the projection operator P_h.
We now assume the following a priori error estimates.

Assumption 1 *For an arbitrary $v \in H_0^2(\Omega) \cap H^4(\Omega)$, there exists a constant C_0 such that*

$$\|\Delta v - \Delta P_h v\|_{L^2} \;\; \leq \;\; C_0 h^2 \|\Delta^2 v\|_{L^2}.$$

Our main purpose of this paper is to find an a priori constant C_0 in the assumption 1 by using guaranteed numerical computations on computer. And as a bi-product of the arguments, we also show a method to get an a posteriori error bound for the approximate solution of the equation (1.1). In the numerical verification method of solutions for two dimensional Navier-Stokes problems(e.g., [2]), we need to enclose a solution of bi-harmonic equations with guaranteed error bounds. In such a situation, the above constant and a posteriori error estimates for the finite element approximation play an essential and important role. The basic techniques used in the below are extension of the method in [5] or [3] to the bi-harmonic problem.

1.3 Preliminary results

We first introduce the following known results.

Lemma 1 *[4] For an arbitrary $\psi \in H_0^2(\Omega) \cap H^4(\Omega)$, it follows that*

$$\|D_x^2 \psi - D_x^2 P_2^x \psi\|_{L^2} \;\; \leq \;\; C h^2 \|D_x^4 \psi\|_{L^2}, \tag{1.2}$$

where the constant C can be taken as $C = 1/\pi^2$. Moreover, the estimate (1.2) is equivalent to the following inequality:

$$\|\psi - P_2^x \psi\|_{L^2} \;\; \leq \;\; C h^2 \|D_x^2 \psi - D_x^2 P_2^x \psi\|_{L^2}.$$

We now show the following inverse inequality for later use.

Lemma 2 *For $\psi_h \in S_h$, it follows that*

$$\|D_x^2 \psi_h\|_{L^2} \;\; \leq \;\; \frac{\kappa}{h^2} \|\psi_h\|_{L^2},$$

where $\kappa = 20\sqrt{21} < 91.6516$.

Proof : Note that it is sufficient to prove the concerning inequality only for one dimensional polynomial of degree 3 on the interval $[0,h]$. In [4], base functions for x-direction on $[0,h]$ are given by

$$\begin{array}{ll} \varphi_1(x) = (x-h)^2(2x+h)/h^3, & \varphi_2(x) = x^2(3h-2x)/h^3, \\ \varphi_3(x) = x(x-h)^2/h^2, & \varphi_4(x) = x^2(x-h)/h^2, \end{array}$$

for $x \in [0,h]$. Hence, setting $\varphi_h := a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + a_4\varphi_4(x)$ and using some guaranteed computations of eigenvalue bounds of a matrix, we obtain

$$\begin{aligned} & \frac{\|D_x^2\varphi_h\|^2_{L^2(0,h)}}{\|\varphi_h\|^2_{L^2(0,h)}} \\ = & \frac{840}{h^4}\left[\vec{a}^T\begin{pmatrix} 6 & -6 & 3 & 3 \\ -6 & 6 & -3 & -3 \\ 3 & -3 & 2 & 1 \\ 3 & -3 & 1 & 2 \end{pmatrix}\vec{a}\right]\cdot\left[\vec{a}^T\begin{pmatrix} 156 & 54 & 22 & -13 \\ 54 & 156 & 13 & -22 \\ 22 & 13 & 4 & -3 \\ -13 & -22 & -3 & 4 \end{pmatrix}\vec{a}\right]^{-1} \\ \leq & \frac{8400}{h^4}, \end{aligned}$$

where $\vec{a} = (a_1, a_2, a_3h, a_4h)^T$.
Here, in order to get the above bound, we used a direct calculation of the matrix eigenvalue. That is, denoting the first and the second matrices in the above by A and B, respectively, let $B = D^TD$ a Cholesky decomposition of B. Then, it is readily seen that the maximum eigenvalue of the symmetric matrix $D^{-T}AD^{-1}$ presents a desired bound. By using a computer algebra system, we confirmed that it is equal to 10. Thus, we can take κ as $\kappa = \sqrt{8400} = 20\sqrt{21}$. ∎

2 Main Results

In this section, we show the constructive a priori and a posteriori error estimations for approximate solutions of the equation (1.1), which are equivalent to the same error estimates for H_0^2-projection of the exact solution.

Let define

$$v_h := P_hv \equiv \sum_{i=1}^{n} v_i\phi_i \in S_h \qquad \text{and} \qquad \overline{\Delta v_h} := P_0\Delta v_h \equiv \sum_{i=1}^{n^*} a_i\phi_i^* \in S_h^*,$$

for an arbitrary $v \in H_0^2(\Omega) \cap H^4(\Omega)$.
Then, $\overline{\Delta v_h}$ satisfies

$$\begin{aligned} (\overline{\Delta v_h}, \phi_h^*)_{L^2} &= (\Delta v_h, \phi_h^*)_{L^2}, \\ &= -(\nabla v_h, \nabla\phi_h^*)_{L^2}, \qquad \forall\phi_h^* \in S_h^*. \end{aligned} \tag{2.1}$$

And, for $g \equiv \Delta^2 v \in L^2(\Omega)$, we set

$$g_h \equiv \sum_{i=1}^{n} g_i \phi_i \in S_h,$$

so that

$$(g_h, \phi_h)_{L^2} = (g, \phi_h)_{L^2}, \qquad \forall \phi_h \in S_h. \tag{2.2}$$

Moreover, we denote some matrices and vectors as

$$\begin{array}{lllll} A_* = (A^*_{ij}) = (\Delta\phi^*_j, \Delta\phi^*_i)_{L^2} & \in R^{n^* \times n^*}, & A = (A_{ij}) = (\Delta\phi_j, \Delta\phi_i)_{L^2} & \in R^{n \times n}, \\ L_* = (L^*_{ij}) = (\phi^*_j, \phi^*_i)_{L^2} & \in R^{n^* \times n^*}, & L = (L_{ij}) = (\phi_j, \phi_i)_{L^2} & \in R^{n \times n}, \\ M = (M_{ij}) = (\nabla\phi_j, \nabla\phi^*_i)_{L^2} & \in R^{n^* \times n}, & N = (N_{ij}) = (\nabla\phi^*_j, \nabla\phi_i)_{L^2} & \in R^{n \times n^*}. \end{array}$$

and

$$\vec{a} = (a_1, \cdots a_{n^*})^T \in R^{n^*}, \quad \vec{v} = (v_1, \cdots v_n)^T, \ \vec{g} = (g_1, \cdots g_n)^T \in R^n.$$

Notice that $\|g\|^2_{L^2} = \|g_h\|^2_{L^2} + \|g - g_h\|^2_{L^2}$. And, when we define the matrix $Q \in R^{n \times n}$ such that $L = QQ^T$, it follows that$\|Q^T \vec{g}\|_E = \|g_h\|_{L^2}$, where $\| \cdot \|_E$ means the Euclidean norm in R^n. Under the above notations, the functions v_h and $\overline{\Delta v_h}$ are determined by solving the following matrix equations:

$$A\vec{v} = L\vec{g},$$

and

$$L_*\vec{a} = -M\vec{v},$$

respectively. Then, we have the following estimates.

Lemma 3 *It follows that*

$$\begin{aligned} \|\Delta v_h - \overline{\Delta v_h}\|_{L^2} &\le \mathbf{X}\|g_h\|_{L^2}, \\ \|g_h - \Delta\overline{\Delta v_h}\|_{L^2} &\le \mathbf{Y}\|g_h\|_{L^2}, \end{aligned}$$

where $\mathbf{X} \equiv \|Q^{-1}XQ^{-T}\|^{1/2}_E$, $\mathbf{Y} \equiv \|Q^{-1}YQ^{-T}\|^{1/2}_E$. *Here, setting* $Z \equiv L^{-1}_* MA^{-1}L$,

$$\begin{aligned} X &\equiv LA^{-1}L - Z^T L_* Z, \\ Y &\equiv L - NZ - Z^T N^T + Z^T A_* Z. \end{aligned}$$

Proof : First, for the estimate $\|\Delta v_h - \overline{\Delta v_h}\|_{L^2}$, we have

$$\begin{aligned} \|\Delta v_h - \overline{\Delta v_h}\|^2_{L^2} &= \|\Delta v_h\|^2_{L^2} - \|\overline{\Delta v_h}\|^2_{L^2} \\ &= \vec{v}^T A\vec{v} - \vec{a}^T L_*\vec{a} \\ &= \vec{v}^T A\vec{v} - \vec{v}^T M^T L^{-1}_* M\vec{v} \\ &= \vec{g}^T (LA^{-1}L - LA^{-1}M^T L^{-1}_* MA^{-1}L)\vec{g}. \end{aligned}$$

Next, for the estimate $\|g_h - \Delta\overline{\Delta v_h}\|_{L^2}$, it follows that

$$
\begin{aligned}
& \|g_h - \Delta\overline{\Delta v_h}\|_{L^2}^2 \\
= & \ (g_h, g_h)_{L^2} - (g_h, \Delta\overline{\Delta v_h})_{L^2} - (\Delta\overline{\Delta v_h}, g_h)_{L^2} + (\Delta\overline{\Delta v_h}, \Delta\overline{\Delta v_h})_{L^2} \\
= & \ (g_h, g_h)_{L^2} + (\nabla g_h, \nabla\overline{\Delta v_h})_{L^2} + (\nabla\overline{\Delta v_h}, \nabla g_h)_{L^2} + (\Delta\overline{\Delta v_h}, \Delta\overline{\Delta v_h})_{L^2} \\
= & \ \vec{g}^T L\vec{g} + \vec{g}^T N\vec{a} + \vec{a}^T N^T\vec{g} + \vec{a}^T A_*\vec{a} \\
= & \ \vec{g}^T L\vec{g} - \vec{g}^T N L_*^{-1} M\vec{v} - \vec{v}^T M^T L_*^{-1} N^T\vec{g} + \vec{a}^T M^T L_*^{-1} A_* L_*^{-1} M\vec{v} \\
= & \ \vec{g}^T (L - N L_*^{-1} M A^{-1} L - L A^{-1} M^T L_*^{-1} N^T + L A^{-1} M^T L_*^{-1} A_* L_*^{-1} M A^{-1} L)\vec{g}.
\end{aligned}
$$

Thus by the above definitions of matrices X, Y and Z, we have

$$
\begin{aligned}
\|\Delta v_h - \overline{\Delta v_h}\|_{L^2}^2 &= \vec{g}^T X\vec{g}, \\
\|g_h - \Delta\overline{\Delta v_h}\|_{L^2}^2 &= \vec{g}^T Y\vec{g},
\end{aligned}
$$

which prove the lemma taking account that $\|Q^T\vec{g}\|_E = \|g_h\|_{L^2}$. ∎

Lemma 4 *For an arbitrary $\psi \in H_0^2(\Omega)$, it follows that*

$$
\|\Delta\psi - \Delta P_0^x P_0^y \psi\|_{L^2} \leq K\|\Delta\psi\|_{L^2},
$$

where $K = \left(2 + 2\,(C\kappa + 1)^2\right)^{1/2}$, the constants C and κ are the same as in Lemma 1 and 2, respectively.

Proof : From the lemma 1 and 2, it follows that

$$
\begin{aligned}
& \|D_x^2(\psi - P_2^x\psi) - D_x^2(P_0^x\psi - P_0^x P_2^x\psi)\|_{L^2} \\
\leq & \ \|D_x^2(\psi - P_2^x\psi)\|_{L^2} + \|D_x^2(\ P_0^x(\psi - P_0^x P_2^x\psi)\ )\|_{L^2} \\
\leq & \ \|D_x^2\psi\|_{L^2} + \frac{\kappa}{h^2}\|P_0^x(\psi - P_2^x\psi)\|_{L^2} \\
\leq & \ \|D_x^2\psi\|_{L^2} + \frac{\kappa}{h^2}\|\psi - P_2^x\psi\|_{L^2} \\
\leq & \ \|D_x^2\psi\|_{L^2} + Ch^2\frac{\kappa}{h^2}\|D_x^2\psi\|_{L^2} \\
\leq & \ (C\kappa + 1)\,\|D_x^2\psi\|_{L^2}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\|D_x^2(\psi - P_0^x P_0^y\psi)\|_{L^2}^2 &= \|D_x^2(\psi - P_0^y\psi) + D_x^2(P_0^y\psi - P_0^y P_0^x\psi)\|_{L^2}^2 \\
&= \|D_x^2(\psi - P_0^y\psi)\|_{L^2}^2 + \|D_x^2(\ P_0^y(\psi - P_0^x\psi)\ )\|_{L^2}^2 \\
&\leq \|D_x^2\psi\|_{L^2}^2 + \|D_x^2(\psi - P_0^x\psi)\|_{L^2}^2 \\
&\leq \|D_x^2\psi\|_{L^2}^2 + \|D_x^2(\psi - P_2^x\psi) - D_x^2(P_0^x\psi - P_0^x P_2^x\psi)\|_{L^2}^2 \\
&\leq \left(1 + (C\kappa + 1)^2\right)\|D_x^2\psi\|_{L^2}^2,
\end{aligned}
$$

where we have used the result just above to obtain the last right-hand side. Similarly, it follows that

$$
\|D_y^2\psi - D_y^2 P_0^x P_0^y\psi\|_{L^2} \leq \left(1 + (C\kappa + 1)^2\right)^{1/2}\|D_y^2\psi\|_{L^2}.
$$

Hence, we have the following estimation:

$$\begin{aligned}\|\Delta\psi - \Delta P_0^x P_0^y \psi\|_{L^2} &\le \|D_x^2(\psi - P_0^x P_0^y\psi)\|_{L^2} + \|D_y^2(\psi - P_0^x P_0^y\psi)\|_{L^2} \\ &\le \left(1 + \left(C^2\kappa + 1\right)^2\right)^{1/2} (\|D_x^2\psi\|_{L^2} + \|D_y^2\psi\|_{L^2}) \\ &\le \left(2 + 2\left(C\kappa + 1\right)^2\right)^{1/2} \left(\|D_x^2\psi\|_{L^2}^2 + \|D_y^2\psi\|_{L^2}^2\right)^{1/2} \\ &\le \left(2 + 2\left(C\kappa + 1\right)^2\right)^{1/2} \|\Delta\psi\|_{L^2},\end{aligned}$$

where, in order to derive the last inequality, we have used the well known equality $\|\Delta\psi\|_{L^2}^2 = ||D_x^2\psi||_{L^2}^2 + ||D_y^2\psi||_{L^2}^2 + 2||D_{xy}\psi||_{L^2}^2$ for any $\psi \in H_0^2(\Omega)$ on an arbitrary domain Ω. Therefore, we obtain the constant K as in the lemma. ∎

Lemma 5 *For an arbitrary* $\psi \in H_0^2(\Omega)$*, it follows that*

$$\|\psi - P_0^x P_0^y \psi\|_{L^2} \ \le\ Ch^2 \|\Delta\psi\|_{L^2},$$

where the constant C *is the same as in Lemma 1.*

Proof : From the lemma 1, it follows that

$$\begin{aligned}\|\psi - P_0^x P_0^y\psi\|_{L^2}^2 &= \|\psi - P_0^x\psi + P_0^x\psi - P_0^x P_0^y\psi\|_{L^2}^2 \\ &= \|\psi - P_0^x\psi\|_{L^2}^2 + \|P_0^x(\psi - P_0^y\psi)\|_{L^2}^2 \\ &\le \|\psi - P_0^x\psi\|_{L^2}^2 + \|\psi - P_0^y\psi\|_{L^2}^2 \\ &\le \|\psi - P_2^x\psi\|_{L^2}^2 + \|\psi - P_2^y\psi\|_{L^2}^2 \\ &\le C^2h^4\|D_x^2\psi\|_{L^2}^2 + C^2h^4\|D_y^2\psi\|_{L^2}^2 \\ &\le C^2h^4\|\Delta\psi\|_{L^2}^2,\end{aligned}$$

which completes the proof. ∎

Now, we show the following two main results of this paper.

Theorem 1 *(constructive a priori error estimates) The constant* C_0 *in Assumption 1 can be taken as*

$$C_0 \ = \ C \cdot \left[\left(K\mathbf{X}/(Ch^2) + \mathbf{Y}\right)^2 + 1\right]^{1/2},$$

where the constants $\mathbf{X}$*,* $\mathbf{Y}$*,* C *and* K *are defined in the previous lemmas 1, 3 and 4.*

Theorem 2 *(a posteriori error estimates) For any* $v \in H_0^2(\Omega) \cap H^4(\Omega)$*, let* $v_h := P_h v \in S_h$ *and* $\overline{\Delta v_h} := P_0 \Delta v_h \in S_h^*$*. Then, it follows that*

$$\|\Delta v - \Delta v_h\|_{L^2} \ \le\ K\|\Delta v_h - \overline{\Delta v_h}\|_{L^2} + Ch^2\|\Delta^2 v - \Delta\overline{\Delta v_h}\|_{L^2},$$

where C *and* K *are defined in the lemmas 1 and 4.*

Proof : (proof of Theorem 1 and 2)
First, for an arbitrary $\psi \in H_0^2(\Omega)$ and $\tilde{\psi}_0 \in S_h$, we have

$$\begin{aligned}
& (\Delta v - \Delta v_h, \Delta\psi)_{L^2} \\
= \; & (\Delta v - \Delta v_h, \Delta\psi - \Delta\tilde{\psi}_0)_{L^2} \\
= \; & (\Delta v - \Delta v_h, \Delta\psi - \Delta\tilde{\psi}_0)_{L^2} + (\overline{\Delta v_h}, \Delta\psi - \Delta\tilde{\psi}_0)_{L^2} - (\Delta\overline{\Delta v_h}, \psi - \tilde{\psi}_0)_{L^2} \\
= \; & (\overline{\Delta v_h} - \Delta v_h, \Delta\psi - \Delta\tilde{\psi}_0)_{L^2} + (\Delta^2 v - \Delta\overline{\Delta v_h}, \psi - \tilde{\psi}_0)_{L^2} \\
\leq \; & \|\Delta v_h - \overline{\Delta v_h}\|_{L^2} \|\Delta\psi - \Delta\tilde{\psi}_0\|_{L^2} + \|\Delta^2 v - \Delta\overline{\Delta v_h}\|_{L^2} \|\psi - \tilde{\psi}_0\|_{L^2}.
\end{aligned}$$

Thus, setting $\psi := v - v_h \in H_0^2(\Omega)$ and $\tilde{\psi}_0 \equiv P_0^x P_0^y \psi \in S_h$, from the lemmas 4 and 5, we obtain the desired estimates in Theorem 2.
Next, using Theorem 2, Lemma 3 and the property of the L^2-projection, it follows that

$$\begin{aligned}
\|\Delta v - \Delta v_h\|_{L^2} & \leq K\|\Delta v_h - \overline{\Delta v_h}\|_{L^2} + Ch^2 \|g - \Delta\overline{\Delta v_h}\|_{L^2} \\
& \leq K\|\Delta v_h - \overline{\Delta v_h}\|_{L^2} + Ch^2 \left(\|g_h - \Delta\overline{\Delta v_h}\|_{L^2} + \|g - g_h\|_{L^2} \right) \\
& \leq \left(K\mathbf{X} + Ch^2\mathbf{Y} \right) \|g_h\|_{L^2} + Ch^2 \|g - g_h\|_{L^2} \\
& \leq \left[\left(K\mathbf{X} + Ch^2\mathbf{Y} \right)^2 + C^2h^4 \right]^{1/2} \|g\|_{L^2} \\
& = \left[\left(K\mathbf{X}/h^2 + C\mathbf{Y} \right)^2 + C^2 \right]^{1/2} h^2 \|g\|_{L^2},
\end{aligned}$$

which immediately completes the proof of Theorem 1. ∎

3 Numerical examples

In this section, we present some numerical examples of a priori and a posteriori error estimates for the approximation of the bi-harmonic problem (1.1). Since the finite element solution $u_h \in S_h$ of (1.1) is defined by

$$(\Delta u_h, \Delta\phi_h)_{L^2} = (f, \phi_h)_{L^2} \quad \forall \phi_h \in S_h,$$

we have $u_h = P_h u$, and thus the above arguments can be applied to the error estimates for this approximate solution u_h. That is, using the procedure in the previous section to define $\overline{\Delta u_h} \equiv P_0 \Delta u_h \in S_h^*$, we obtain the a priori and a posteriori error estimates of the form

$$\|\Delta u - \Delta u_h\|_{L^2} \leq C_0 h^2 \|f\|_{L^2}, \tag{3.1}$$

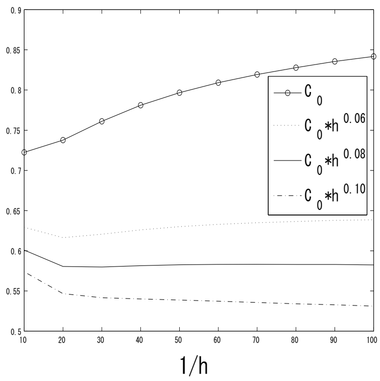
and

$$\|\Delta u - \Delta u_h\|_{L^2} \leq K\|\Delta u_h - \overline{\Delta u_h}\|_{L^2} + Ch^2 \|f - \Delta\overline{\Delta u_h}\|_{L^2}, \tag{3.2}$$

respectively. Here, constants C_0, K and C are same as in theorems in Section 2.
We first show several computational results for the constructive a priori constants in Theorem 1 by Table 1.

Table 1: Numerical results for the a priori constant

$1/h$	C_0	$\mathbf{X}$	$\mathbf{Y}$	C_0h^2	C_0/C
10	0.7225	3.6718e-4	1.7641	7.2256e-3	7.1314
20	0.7377	9.3350e-5	1.8257	1.8443e-3	7.2812
30	0.7611	4.3440e-5	1.8057	8.4573e-4	7.5123
40	0.7811	2.5416e-5	1.7780	4.8819e-4	7.7093
50	0.7967	1.6764e-5	1.7536	3.1868e-4	7.8633
60	0.8091	1.1918e-5	1.7341	2.2477e-4	7.9862
70	0.8193	8.9195e-6	1.7195	1.6720e-4	8.0864
80	0.8278	6.9315e-6	1.7095	1.2934e-4	8.1703
90	0.8356	5.5486e-6	1.7036	1.0317e-4	8.2478
100	0.8418	4.5390e-6	1.7006	8.4185e-5	8.3087



The constant C in the table is taken as $C = 1/\pi^2$.

Next, we present some examples of the a posteriori error for the following bi-harmonic problem.

$$\begin{aligned} \Delta^2 u &= f && \text{in} \quad \Omega, \\ u &= \partial_n u = 0 && \text{on} \quad \partial\Omega, \end{aligned} \tag{3.3}$$

where $\Omega = (0,1)^2$ and

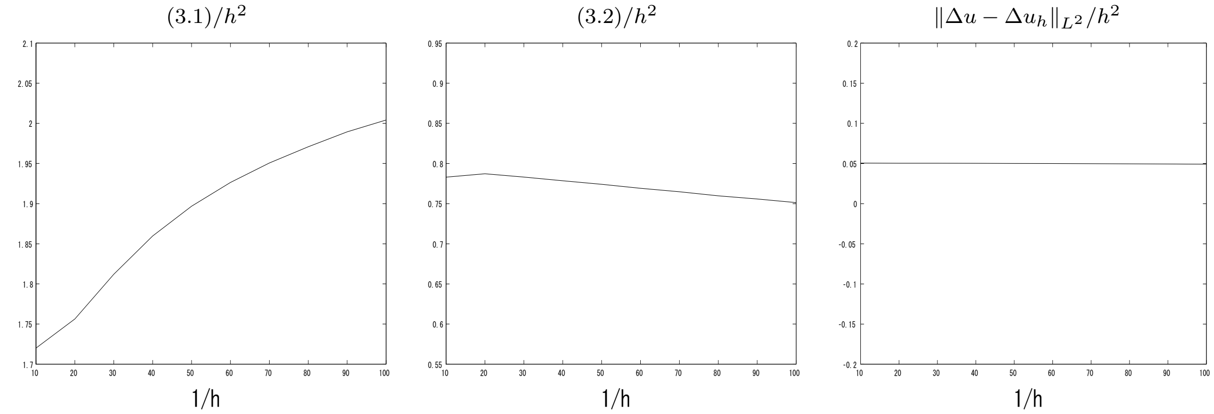
$$f \equiv f(x,y) = 8\left(3x^2(1-x)^2 + xy(x-1)(y-1)(2x-1)(2y-1) + 3y^2(1-y)^2\right).$$

The exact solution of (3.3) is given by $u \equiv u(x,y) = x^2y^2(1-x)^2(1-y)^2$.

Table 2 shows numerical results for the a priori and a posteriori error estimates in (3.1) and (3.2), respectively. Note that $\|f\|_{L^2} = \sqrt{992/175} < 2.3809$.

Table 2: Numerical results for the a priori and a posteriori estimates in (3.1) and (3.2)

$1/h$	(3.1)	(3.2)	$\|\Delta u - \Delta u_h\|_{L^2}$	$\|\Delta u_h - \overline{\Delta u_h}\|_{L^2}$	$\|f - \Delta\overline{\Delta u_h}\|_{L^2}$
10	1.7201e-2	7.8301e-3	5.0527e-4	4.4700e-4	1.2798
20	4.3909e-3	1.9680e-3	1.2595e-4	1.1897e-4	0.9047
30	2.0134e-3	8.7011e-4	5.5883e-5	5.3843e-5	0.7385
40	1.1623e-3	4.8659e-4	3.1376e-5	3.0521e-5	0.6394
50	7.5873e-4	3.0962e-4	2.0034e-5	1.9598e-5	0.5716
60	5.3510e-4	2.1363e-4	1.3863e-5	1.3611e-5	0.5215
70	3.9809e-4	1.5609e-4	1.0154e-5	9.9966e-6	0.4826
80	3.0795e-4	1.1870e-4	7.7390e-6	7.6331e-6	0.4511
90	2.4561e-4	9.3310e-5	6.0946e-6	6.0203e-6	0.4250
100	2.0042e-4	7.5133e-5	4.9152e-6	4.8611e-6	0.4029



All the computations were carried out by MATLAB on the Dell Precision 650 Workstation (Intel Xeon Dual CPU 3.20GHz).

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