A computational approach to constructive a priori error estimate for finite element approximations of bi-harmonic problems in nonconvex polygonal domains

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Abstract

In the numerical verification method of solutions for nonlinear fourth order elliptic equations in nonconvex polygonal domains, it is important to find a constant in the constructive a priori error estimate for the finite element approximation of bi-harmonic problems. We show these procedures by verified computational techniques using the Hermite spline functions for the two dimensional L-shaped domain. Several numerical examples which confirm the actual effectiveness of the method are presented.

Keywords : bi-harmonic problems, nonconvex polygonal domains, constructive a priori error estimate

1 Introduction

In this paper, we consider the guaranteed error bounds of the finite element approximations for the following equation:

$$\begin{array}{rclcll} \Delta^2 u & = & f & & \text{in} & \Omega, \\ u & = & \partial_n u & = 0 & \text{on} & \partial\Omega, \end{array} \tag{1.1}$$

where $f \in L^2(\Omega)$ with a nonconvex polygonal domain Ω in R^2, and $\partial_n u$ denotes the outer normal derivative of u. In particular, we consider the case of L-shaped domains.

1.1 Notations

In the below, setting $\Omega := (-1,1)^2 \backslash [-1,0]^2$ for simplicity, we denote the L^2 inner product on Ω by $(\cdot,\cdot)_{L^2}$ and the norm by $\|\cdot\|_{L^2}$. We also denote the usual k-th order L^2 Sobolev space on Ω by $H^k(\Omega)$ for any positive integer k as well as the spaces $H_0^2(\Omega)$ and $X(\Omega)$ are defined by

$$\begin{array}{rcl} H_0^2(\Omega) & := & \left\{ v \in H^2(\Omega) : \ v = \partial_n v = 0 \text{ on } \partial\Omega \right\}, \\ X(\Omega) & := & \left\{ v \in H_0^2(\Omega) : \ \Delta^2 v \in L^2(\Omega) \right\}, \end{array}$$

respectively.

Let Ω_* be a rectangular domain which includes Ω. We take Ω_{out} as a residual domain such that $\Omega_* = \Omega \cup \Omega_{\text{out}}$ (see Figure 1 and [6]).

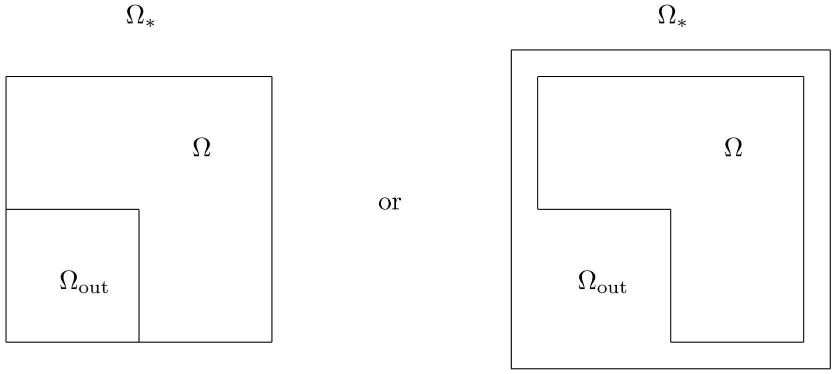


Figure 1: Convex extension of the domain

Then, we extend an arbitrary $v \in H_0^2(\Omega)$ to the function on Ω_* with $v = \partial_n v = 0$ in Ω_{out}, which belongs to $H_0^2(\Omega_*)$ and is also denoted by v.

Let $S_h \subset H_0^2(\Omega)$ be a finite element subspace, which is spanned by two dimensional Hermite spline functions $\{\phi_i\}_{1\le i\le n}$ with homogeneous boundary conditions in H_0^2 sense(e.g., [5]). Moreover, we define the space S_h^* which is spanned by the basis $\{\phi_i^*\}_{1\le i\le n^*}$, as the finite element subspace of $H_0^2(\Omega_*)$ satisfying $S_h \subset S_h^*$. Namely, the set of base functions $\{\phi_i^*\}_{1\le i\le n^*}$ consists of the elements in $\{\phi_i\}_{1\le i\le n}$ and the Hermite spline functions corresponding to the residual domains(*cf.* [6]). Note that the finite element subspaces S_h and S_h^* are dependent on the parameter h.

Next, we define the H_0^2-projection $P_h : H_0^2(\Omega) \to S_h$ of $v \in H_0^2(\Omega)$ by

$$(\Delta v - \Delta P_h v, \Delta \phi_h)_{L^2} = 0, \quad \forall \phi_h \in S_h.$$

The H_0^2-projection $P_h^* : H_0^2(\Omega_*) \to S_h^*$ is analogously defined(see Figure 2).

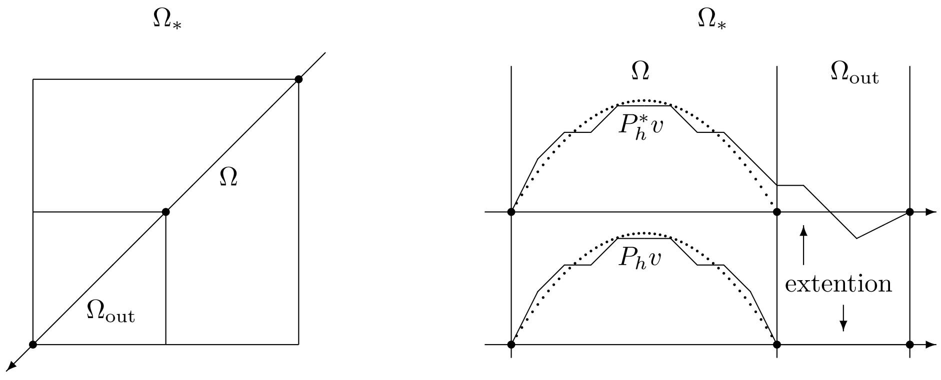


Figure 2: An image of $P_h v$ and $P_h^* v$ for $v \in H_0^2(\Omega)$

Notice that for all notations, we sometimes denote the notation with Ω when it depends on Ω.

1.2 Motivation

Let $u_h \in S_h$ be an approximate solution of (1.1) satisfying

$$(\Delta u_h, \Delta \phi_h)_{L^2} = (f, \phi_h)_{L^2} \quad \forall \phi_h \in S_h.$$

Then, note that we have $u_h = P_h u$ by the definition. Also, on the regularity of solutions, note that the solution u of (1.1) belongs to $H_0^2(\Omega) \cap H^{5/2+\alpha}(\Omega)$, where $0 \le \alpha \le 3/2$ which depends on the shape of the domains(e.g., [1][2]). Particularly, in case that $\Omega = (-1,1)^2 \backslash [-1,0]^2$, we have $\alpha = 0$. Therefore, in what follows, we consider the actual order of magnitude for the projection error, which is our main interest in the present paper.
We now assume the following a priori error estimates for the projection operator P_h.

Assumption 1 *For an arbitrary $v \in X(\Omega)$, there exists a constant $C(h)$ dependent on h such that*

$$\|\Delta v - \Delta P_h v\|_{L^2} \quad \le \quad C(h)\|\Delta^2 v\|_{L^2}.$$

Our main purpose of this paper is to find an a priori constant $C(h)$ in the assumption 1 by using guaranteed numerical computations on computer. In the numerical verification method of solutions for two dimensional Navier-Stokes problems(e.g., [3]), we need to enclose a solution of bi-harmonic equations with guaranteed error bounds. In such a situation, the above constant and a posteriori error estimates for the finite element approximation play an essential and important role. The basic techniques used in the below are extension of the method in [6] to the bi-harmonic problem.

1.3 Preliminary results

We first introduce the following known results.

Theorem 1 *For an arbitrary $v \in X(\Omega)$, Assumption 1 is equivalent to the following inequality:*

$$\|v - P_h v\|_{L^2} \quad \le \quad C(h)\|\Delta v - \Delta P_h v\|_{L^2}. \tag{1.2}$$

Proof : First, we assume that Assumption 1 holds. Let $\phi \in X(\Omega)$ be a solution of the following bi-harmonic equation:

$$\begin{array}{rclcl} \Delta^2 \phi & = & v - P_h v & & \text{in} \quad \Omega, \\ \phi & = & \partial_n \phi & = \quad 0 & \text{on} \quad \partial\Omega. \end{array}$$

Then, from $(\Delta v - \Delta P_h v, \Delta P_h \phi)_{L^2} = 0$, it follows that

$$\begin{aligned} \|v - P_h v\|_{L^2}^2 = (v - P_h v, v - P_h v)_{L^2} \quad &= \quad (\Delta\phi, \Delta v - \Delta P_h v)_{L^2} \\ &= \quad (\Delta\phi - \Delta P_h \phi, \Delta v - \Delta P_h v)_{L^2} \\ &\le \quad \|\Delta\phi - \Delta P_h \phi\|_{L^2}\|\Delta v - \Delta P_h v\|_{L^2}. \end{aligned}$$

Thus, using Assumption 1, we can obtain

$$\|v - P_h v\|_{L^2} \le C(h)\|\Delta v - \Delta P_h v\|_{L^2}.$$

Next, we assume that the inequality (1.2) holds. Then, from the definition of H_0^2-projection, we have

$$\begin{aligned}
\|\Delta v - \Delta P_h v\|_{L^2}^2 &= (\Delta v - \Delta P_h v, \Delta v - \Delta P_h v)_{L^2} \\
&= (\Delta v, \Delta v - \Delta P_h v)_{L^2} \\
&= (\Delta^2 v, v - P_h v)_{L^2} \\
&\leq \|\Delta^2 v\|_{L^2} \|v - P_h v\|_{L^2}.
\end{aligned}$$

Hence, we can obtain $\|\Delta v - \Delta P_h v\|_{L^2} \leq C(h)\|\Delta^2 v\|_{L^2}$. Therefore, this proof is completed. ∎

Theorem 2 *[2][4] For an arbitrary $v \in X(\Omega_*)$, it follows that $v \in H_0^2(\Omega_*) \cap H^4(\Omega_*)$. Then, there exists a constant C_0 such that*

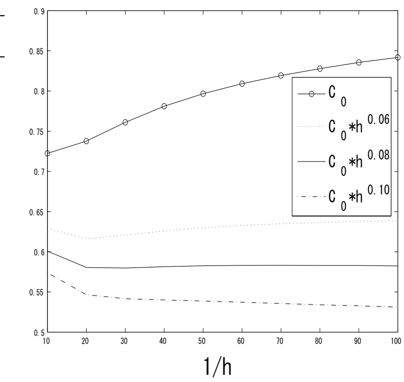
$$\|\Delta v - \Delta P_h^* v\|_{L^2(\Omega_*)} \leq C_0 h^2 \|\Delta^2 v\|_{L^2(\Omega_*)},$$

where C_0 is numerically determined.

For the constant C_0 in Theorem 2, the following numerical results are obtained(refer [4]).

Table 1: Numerical results for the a priori constant

$1/h$	C_0	C_0h^2
10	0.7225	7.2256e-3
20	0.7377	1.8443e-3
30	0.7611	8.4573e-4
40	0.7811	4.8819e-4
50	0.7967	3.1868e-4
60	0.8091	2.2477e-4
70	0.8193	1.6720e-4
80	0.8278	1.2934e-4
90	0.8356	1.0317e-4
100	0.8418	8.4185e-5



Remark : Theorem 2 is equivalent to the following inequality:

$$\|v - P_h^* v\|_{L^2(\Omega_*)} \leq C_0 h^2 \|\Delta v - \Delta P_h^* v\|_{L^2(\Omega_*)}.$$

This remark is proven by the process similar to Theorem 1.

2 A computational procedure for $C(h)$

For the numerical verification method, it is important to obtain the constructive a priori error estimate between a function and its projection. However, it is generally known that, in the case of nonconvex polygonal domains, the lower regularity of the solution leads the lower accuracy in this estimate. Hence, using techniques in [6], we show a computational procedure of the constructive a priori error estimation for the present case. Namely, our aim is to show how to evaluate the a priori constant $C(h)$ in the current situation.

We first define the constant K as follows:

$$K \equiv \sup_{v \in H_0^2(\Omega)} \frac{\|P_h^* v - P_h v\|_{L^2(\Omega)}}{\|P_h^* v - P_h v\|_{L^2(\Omega_{\mathrm{out}})}}. \tag{2.1}$$

Then, we have the following main result of this paper.

Theorem 3 *For the constant $C(h)$ in Assumption 1, it holds that*

$$C(h) \leq C_0 h^2 \sqrt{1 + K^2},$$

where the constant C_0 is the same as in Theorem 2.

Proof : For an arbitrary $v \in X(\Omega)$, it follows that

$$\|v - P_h^* v\|_{L^2(\Omega_*)}^2 = \|v - P_h^* v\|_{L^2(\Omega)}^2 + \|P_h^* v\|_{L^2(\Omega_{\mathrm{out}})}^2.$$

Thus, for $0 \leq \theta \leq \pi/2$, we have the following equality:

$$\begin{aligned} \|v - P_h^* v\|_{L^2(\Omega)} &= \|v - P_h^* v\|_{L^2(\Omega_*)} \cos\theta, \\ \|P_h^* v\|_{L^2(\Omega_{\mathrm{out}})} &= \|v - P_h^* v\|_{L^2(\Omega_*)} \sin\theta. \end{aligned}$$

Moreover, using Remark of Theorem 2 and the above, we have

$$\begin{aligned} \|v - P_h^* v\|_{L^2(\Omega)} &\leq C_0 h^2 \|\Delta v - \Delta P_h^* v\|_{L^2(\Omega_*)} \cos\theta, \\ \|P_h^* v\|_{L^2(\Omega_{\mathrm{out}})} &\leq C_0 h^2 \|\Delta v - \Delta P_h^* v\|_{L^2(\Omega_*)} \sin\theta. \end{aligned}$$

Hence, using K in (2.1) and the fact that $\Omega \subseteq \Omega_*$, we obtain

$$\begin{aligned} \|v - P_h v\|_{L^2(\Omega)} &= \|v - P_h^* v + P_h^* v - P_h v\|_{L^2(\Omega)} \\ &\leq \|v - P_h^* v\|_{L^2(\Omega)} + \|P_h^* v - P_h v\|_{L^2(\Omega)} \\ &\leq C_0 h^2 (\cos\theta + K \sin\theta) \|\Delta v - \Delta P_h^* v\|_{L^2(\Omega_*)} \\ &\leq C_0 h^2 \sqrt{1 + K^2} \|\Delta v - \Delta P_h^* v\|_{L^2(\Omega_*)}. \end{aligned}$$

Since

$$\begin{aligned} \|\Delta v - \Delta P_h^* v\|_{L^2(\Omega_*)}^2 &= \|\Delta(v - P_h v) - \Delta P_h^*(v - P_h v)\|_{L^2(\Omega_*)}^2 \\ &= \|\Delta(v - P_h v)\|_{L^2(\Omega_*)}^2 - \|\Delta P_h^*(v - P_h v)\|_{L^2(\Omega_*)}^2 \\ &= \|\Delta v - \Delta P_h v\|_{L^2(\Omega)}^2 - \|\Delta P_h^*(v - P_h v)\|_{L^2(\Omega_*)}^2, \end{aligned}$$

it follows that $\|v - P_h v\|_{L^2(\Omega)} \leq C_0 h^2 \sqrt{1 + K^2} \|\Delta v - \Delta P_h v\|_{L^2(\Omega)}$. Therefore, this proof is completed from Theorem 1 and the definition of K. ∎

3 The eigenvalue problems for the constant K

In the section, we introduce a method on the computation of the constant K which is defined in the previous section. We now introduce some notations below.

First, we define three $n^* \times n^*$ matrices $H = (H_{ij})$, $L = (L_{ij})$ and $D = (D_{ij})$ as follows.

$$H_{ij} = (\Delta\phi_j^*, \Delta\phi_i^*)_{L^2(\Omega_*)}, \quad L_{ij} = (\phi_j^*, \phi_i^*)_{L^2(\Omega)}, \quad D_{ij} = (\phi_j^*, \phi_i^*)_{L^2(\Omega_{\text{out}})},$$

for $1 \le i, j \le n^*$. And, we take $B = \{J_1, \cdots, J_b\}$ as a set of the numbers corresponding to nodes on the common boundary of Ω and Ω_{out}(i.e. $\partial\Omega\cap\partial\Omega_{\text{out}}$).

Then, for an arbitrary $v \in H_0^2(\Omega)$, letting $v_h \equiv P_h v$ and $v_h^* \equiv P_h^* v$, it follows that

$$\begin{array}{rcll} (\Delta v_h^* - \Delta v_h, \Delta\phi_i^*)_{L^2(\Omega_*)} & = & (\Delta v - \Delta v_h, \Delta\phi_i^*)_{L^2(\Omega)} & \text{if } i \in B, \\ (\Delta v_h^* - \Delta v_h, \Delta\phi_i^*)_{L^2(\Omega_*)} & = & 0 & \text{otherwise.} \end{array}$$

Here, we define a vector $\mathbf{g} = \big(\mathbf{g}_i\big) \in R^{n^*}$ by

$$\mathbf{g}_i = \begin{cases} (\Delta v - \Delta v_h, \Delta\phi_i^*)_{L^2(\Omega)} & \text{if } i \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the norms $\|v_h^* - v_h\|_{L^2(\Omega)}$ and $\|v_h^* - v_h\|_{L^2(\Omega_{\text{out}})} = \|v_h^*\|_{L^2(\Omega_{\text{out}})}$ can be written as

$$\|v_h^* - v_h\|_{L^2(\Omega)}^2 = \mathbf{g}^T X \mathbf{g}, \quad \|v_h^*\|_{L^2(\Omega_{\text{out}})}^2 = \mathbf{g}^T Y \mathbf{g},$$

where $X := H^{-1}LH^{-1}$ and $Y := H^{-1}DH^{-1}$. Moreover, we define $b \times b$ matrices $\mathbf{X} = (\mathbf{X}_{ij})$ and $\mathbf{Y} = (\mathbf{Y}_{ij})$ by $\mathbf{X}_{ij} = X_{J_iJ_j}$ and $\mathbf{Y}_{ij} = Y_{J_iJ_j}$ for $1 \le i, j \le b$, respectively. Then, the constant K in (2.1) is given by solving the following generalized eigenvalue problem:

$$\mathbf{X}\mathbf{b} = \lambda^2 \mathbf{Y}\mathbf{b}, \tag{3.1}$$

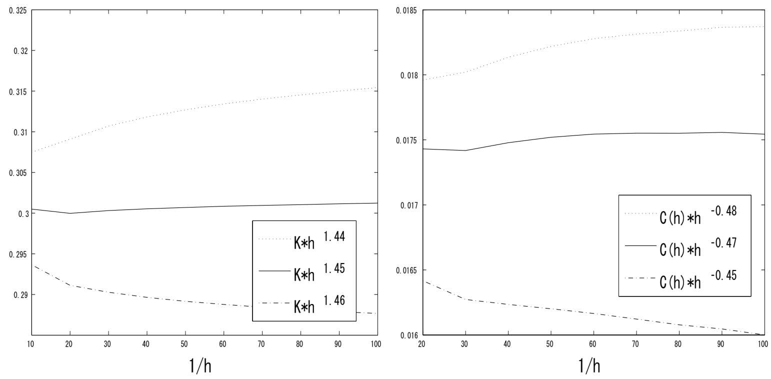
for $\mathbf{b} \in R^b$. Namely, K can be taken as $K = \max_{1\le i\le b} \sqrt{\lambda_i^2}$, where λ_i^2 is the eigenvalue of (3.1). Note that the dimension of this eigenvalue problem is small.

4 Numerical results

In this section, we show several computational results for the actual value of a priori constants $C(h)$ in Theorem 3 by Table 2 below.

Table 2: Numerical results for the a priori constant $C(h)$

$1/h$	10	20	30	40	50
K	8.4690	23.0985	41.6299	63.2237	87.4241
$C(h)$	6.1618e-3	4.2640e-3	3.5217e-3	3.0869e-3	2.7862e-3
$1/h$	60	70	80	90	100
K	113.9375	142.5256	173.0248	205.3179	239.2797
$C(h)$	2.5610e-3	2.3830e-3	2.2379e-3	2.1182e-3	2.0143e-3



In [2], the order of magnitude for the a priori constant $C(h)$ is presented as $C(h) = O(h^{0.5})$ for L-shaped domains. From Table 2 it is seen that our results almost coincide with this order.

All computations in tables are carried out on the Dell Precision 650 Workstation Intel Xeon Dual CPU 3.20GHz by MATLAB.

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